

Threshold Moving Average Models Invertibility

Alessandra Amendola, Marcella Niglio, Cosimo Damiano Vitale

Abstract In this paper we evaluate the invertibility of the Threshold Moving Average model. The results are obtained generalizing some issues given in the literature and introducing a distinction between “local” and “global” invertibility. To better clarify this last difference, theoretical results and empirical examples are jointly considered and discussed.

Key words: Time series analysis, invertibility, nonlinearity, threshold model.

1 Introduction

The study of stochastic processes is often focused on the investigation of the probability properties (such as stationarity, ergodicity and invertibility) that give important requirements under which to make statistical inference and generate predictions. In this domain the most relevant results have been reached in the context of linear models (see [1] among the others) whereas a small number of contributions has been provided for nonlinear structures.

Based on these considerations, in this paper we present the invertibility of nonlinear moving average models. In particular we focus on the so called Threshold Moving Average (TMA) models that belong to the class of threshold nonlinear structures widely presented in [7].

The TMA model has been extensively investigated in [2] whereas [5] propose a

Alessandra Amendola
Di.S.E.S., University of Salerno, e-mail: alamendola@unisa.it

Marcella Niglio
Di.S.E.S., University of Salerno, e-mail: mniglio@unisa.it

Cosimo Damiano Vitale
Di.S.E.S., University of Salerno, e-mail: vitale@unisa.it

quasi-LR test for linear moving average (MA) against TMA models (recently generalized in [4] to the TMA-GARCH models) and give some contribution on the TMA invertibility. This last topic has been empirically treated in [3] and, more recently, it has been examined in [6]. They establish necessary and sufficient conditions for the invertibility of TMA models with moving average order $q = 1$ and $k = 2$ regimes, even depicting some developments for the more general TMA model with $k > 2$.

In the present paper the invertibility of these models is further discussed. In particular we provide sufficient conditions under which a general TMA model of order $q > 1$ with $k > 2$ regimes, is invertible.

More specifically, in Sect. 2 we present the invertibility conditions of a $TMA(k; q)$ model whereas illustrative examples are presented in Sect. 3. These examples show that when the TMA model is under analysis, we can distinguish between “local” and “global” invertibility, or, in other words, a TMA model can be “globally” invertible even in presence of regimes “locally” noninvertible. Some concluding remarks and hints of future research are reported at the end.

2 Invertibility of $TMA(k; q)$ models

Given a strictly stationary process X_t , it follows a $TMA(2; 1)$ model if

$$X_t = \varepsilon_t - [\phi + \psi I(X_{t-d} \leq r)] \varepsilon_{t-1}, \quad (1)$$

where $\{\varepsilon_t\}$ is a sequence of independent and identically distributed (i.i.d.) random variables, $|r| \leq \infty$ is the threshold variable, $d \in \mathcal{N}^+$ is the threshold delay and $I(X_{t-d} \leq r)$ is the indicator function:

$$I(X_{t-d} \leq r) \equiv I_{t-d} = \begin{cases} 1 & \text{if } X_{t-d} \leq r \\ 0 & \text{if } X_{t-d} > r. \end{cases} \quad (2)$$

The statistical properties of model (1) have been investigated in [6]. They give the conditions under which the model admits a unique strictly stationary and ergodic solution and further show, under proper assumptions on ψ and on the distribution of $\{\varepsilon_t\}$, a sufficient condition for its invertibility. In particular they state that given a strictly stationary and ergodic process $X_t \sim TMA(2; 1)$ with $E[|\ln |X_t||] < \infty$, model (1) is invertible if:

$$|\phi|^{1-F_X(r)} |\phi + \psi|^{F_X(r)} < 1, \quad (3)$$

where $0 \leq F_X(r) = P(X_t \leq r) = E[I_{t-d}] = p \leq 1$.¹

Starting from these results, it is interesting to remark:

¹ $F_X(r)$ defines the probability that X_t belongs to the first regime or, in other words $E[I_{t-d}]$; conversely $1 - F_X(r)$ is the probability that X_t belongs to the second regime, or $1 - E[I_{t-d}]$.

Remark 1. When $E[I_{t-d}] = 1$ (or $E[I_{t-d}] = 0$) model (1) degenerates to a linear moving average process (widely discussed in [1]) and the inequality (3) matches the invertibility conditions given for this class of models;

Remark 2. The invertibility region (over the parametric space) (3) is wider than that obtained overlapping the invertibility regions of the two regimes (that, in any case, is included in (3)).

The conditions and the results established on the invertibility of the TMA(2;1) model in Theorem 2 of [6] represent the starting point of the issues given in the following pages where more general results for a TMA($k; q$) model are provided.

To simplify the presentation of the theoretical results without constrain their generality, we show all results for a TMA($k; q$) model, with $k = 2$ and $q \geq 1$.

Let $X_t \sim TMA(2; q)$:

$$X_t = \varepsilon_t - \sum_{i=1}^q [\phi_i + \psi_i I(X_{t-d} \leq r)] \varepsilon_{t-i}, \quad (4)$$

model (4) can be given as

$$\begin{aligned} X_t &= \varepsilon_t - \sum_{i=1}^q (\phi_i + \psi_i) I_{t-d} \varepsilon_{t-i} - \sum_{i=1}^q \phi_i (1 - I_{t-d}) \varepsilon_{t-i} \\ &= [1 - I_{t-d} \Theta(B) - (1 - I_{t-d}) \Phi(B)] \varepsilon_t \end{aligned} \quad (5)$$

where B is the lag operator such that $B^s X_t = X_{t-s}$, $\Phi(B) = \sum_{i=1}^q \phi_i B^i$, $\Theta(B) = \sum_{i=1}^q \theta_i B^i$ and $\theta_i = \phi_i + \psi_i$, for $i = 1, 2, \dots, q$.

The model (4), can be also presented in the following notation. Let:

$$\varepsilon_t = \begin{bmatrix} \varepsilon_t \\ \varepsilon_{t-1} \\ \vdots \\ \varepsilon_{t-q+1} \end{bmatrix}, \quad \mathbf{X}_t = \begin{bmatrix} X_t \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \theta = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_q \end{bmatrix}, \quad \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_q \end{bmatrix}$$

four $(q \times 1)$ vectors and

$$\mathbf{A}_{t-d} = I_{t-d} \begin{bmatrix} \theta' \\ \mathbf{I} \ \mathbf{0} \end{bmatrix} + (1 - I_{t-d}) \begin{bmatrix} \phi' \\ \mathbf{I} \ \mathbf{0} \end{bmatrix} = I_{t-d} \Theta + (1 - I_{t-d}) \Phi,$$

where $\mathbf{0}$ is a $[(q-1) \times 1]$ null vector and $\mathbf{I}$ is a $(q-1)$ identity matrix, the Markovian representation of model (5) is:

$$\varepsilon_t = \mathbf{X}_t + \mathbf{A}_{t-d} \varepsilon_{t-1}. \quad (6)$$

After n iterations, model (6) becomes:

$$\varepsilon_t = \mathbf{X}_t + \sum_{j=1}^n \left(\prod_{i=0}^{j-1} \mathbf{A}_{t-d-i} \right) \mathbf{X}_{t-j} + \left(\prod_{i=0}^n \mathbf{A}_{t-d-i} \right) \varepsilon_{t-n-1},$$

where ε_t , if pre-multiplied by the vector $\mathbf{1}' = (1, 0, \dots, 0)$ such that $\varepsilon_t = \mathbf{1}' \varepsilon_t$, becomes:

$$\varepsilon_t = X_t + \mathbf{1}' \sum_{j=1}^n \left(\prod_{i=0}^{j-1} \mathbf{A}_{t-d-i} \right) \mathbf{X}_{t-j} + \mathbf{1}' \left(\prod_{i=0}^n \mathbf{A}_{t-d-i} \right) \varepsilon_{t-n-1}. \quad (7)$$

Model (7) is used to discuss the invertibility of the TMA(2;q) model as follows.

Theorem 1. *Let $X_t \sim TMA(2;q)$ a strictly stationary and ergodic process with $E[|\ln |X_t||] < \infty$, X_t is invertible if:*

$$|\lambda(\Theta)|^p |\lambda(\Phi)|^{1-p} < 1, \quad (8)$$

where $\lambda(\Theta)$ [or $\lambda(\Phi)$] is the dominant eigenvalue of Θ [or Φ] and $p = E[I_{t-d}]$.

Proof. Starting from model (7), then

$$\varepsilon_t - X_t - \mathbf{1}' \sum_{j=1}^n \left(\prod_{i=0}^{j-1} \mathbf{A}_{t-d-i} \right) \mathbf{X}_{t-j} = \mathbf{1}' \left(\prod_{i=0}^n \mathbf{A}_{t-d-i} \right) \varepsilon_{t-n-1}$$

and X_t is invertible if

$$\prod_{i=0}^n |\mathbf{A}_{t-d-i}| \xrightarrow{p} \mathbf{0}, \quad \text{as } n \rightarrow \infty. \quad (9)$$

Using standard results of matrix algebra and the notation in Appendix, it follows that:

$$\begin{aligned} \prod_{i=0}^n |\mathbf{A}_{t-d-i}| &= \prod_{i=0}^n |I_{t-d-i} \Theta + (1 - I_{t-d-i}) \Phi| \\ &< \prod_{i=0}^n |I_{t-d-i} \lambda(\Theta) \mathbf{I} + (1 - I_{t-d-i}) \lambda(\Phi) \mathbf{I}| \\ &= \exp \left\{ \sum_{i=0}^n \ln |I_{t-d-i} \lambda(\Theta) \mathbf{I} + (1 - I_{t-d-i}) \lambda(\Phi) \mathbf{I}| \right\}. \end{aligned}$$

Recalling that I_{t-d-i} is the indicator function defined in (2), we have:

$$\prod_{i=0}^n |\mathbf{A}_{t-d-i}| < \exp \left\{ \ln |\lambda(\Theta) \mathbf{I}| \sum_{i=0}^n I_{t-d-i} + \ln |\lambda(\Phi) \mathbf{I}| \sum_{i=0}^n (1 - I_{t-d-i}) \right\}.$$

Noting that $\frac{1}{n+1} \sum_{i=0}^n I_{t-d-i} = p + O_p(n^{-1/2})$, then:

$$\begin{aligned}
\prod_{i=0}^n |\mathbf{A}_{t-d-i}| &\prec \exp \left\{ \ln |\lambda(\Theta) \mathbf{I}| (n+1)(p + O_p(n^{-1/2})) \right. \\
&\quad \left. + \ln |\lambda(\Phi) \mathbf{I}| (n+1)(1-p + O_p(n^{-1/2})) \right\} \\
&= |\lambda(\Theta) \mathbf{I}|^{(n+1)(p + O_p(n^{-1/2}))} |\lambda(\Phi) \mathbf{I}|^{(n+1)(1-p + O_p(n^{-1/2}))} \\
&= \left(|\lambda(\Theta)|^p |\lambda(\Phi)|^{1-p} \right)^{(n+1)} |\lambda(\Theta) \lambda(\Phi)|^{(n+1)O_p(n^{-1/2})} \mathbf{I},
\end{aligned}$$

where, observing that $\left(|\lambda(\Theta)|^p |\lambda(\Phi)|^{1-p} \right)^{(n+1)}$ is of order $O(n)$, the convergence (9) is obtained if:

$$|\lambda(\Theta)|^p |\lambda(\Phi)|^{1-p} < 1.$$

□

Under the conditions of Theorem 1, model (7) can be given as:

$$\varepsilon_t = X_t + \sum_{j=1}^{\infty} \mathbf{1}' \left[\prod_{i=0}^{j-1} (\Theta I_{t-d-i} + \Phi(1 - I_{t-d-i})) \right] \mathbf{X}_{t-j}. \quad (10)$$

Remark 3. When $\Theta = \Phi$ (or $p = 0, 1$) in model (6), the inequality (8) corresponds to the invertibility condition of a linear MA(q) model whereas if $q = 1$ then $\lambda(\Theta) = \theta$, $\lambda(\Phi) = \phi$ and the condition (8) matches (3).

The inequality (8) and model (10) can be generalized to the case of a TMA($k; q$) model,

$$X_t = \varepsilon_t + \sum_{s=1}^k (\psi_{s,1} \varepsilon_{t-1} + \psi_{s,2} \varepsilon_{t-2} + \dots + \psi_{s,q} \varepsilon_{t-q}) I_{s,t-d}, \quad (11)$$

where $\psi_{s,i}$ is the parameter i of regime s (for $s = 1, 2, \dots, k$ and $i = 1, 2, \dots, q$), $I_{s,t-d}$ is the indicator function that now becomes:

$$I_{s,t-d} = \begin{cases} 1 & \text{if } r_{s-1} \leq X_{t-d} < r_s \\ 0 & \text{otherwise,} \end{cases}$$

with $\sum_{s=1}^k I_{s,t-d} = 1$ and threshold values $-\infty = r_0 \leq r_1 \leq \dots \leq r_k = \infty$.

Model (11) has markovian form (6) with

$$\mathbf{A}_{t-d} = \sum_{s=1}^k I_{s,t-d} \Psi_s, \quad \text{where } \Psi_s = \begin{bmatrix} \Psi' \\ \mathbf{I} \quad \mathbf{0} \end{bmatrix} \quad \text{and} \quad \Psi' = (\psi_{s,1}, \dots, \psi_{s,q}),$$

from which the invertibility conditions are provided in the following corollary.

Corollary 1. Let $X_t \sim TMA(k; q)$, under the assumptions of Theorem 1, X_t is invertible if:

$$\prod_{s=1}^k |\lambda(\Psi_s)|^{p_s} < 1, \quad \text{with } p_s = E[I_{s,t-d}],$$

and ε_t can be given as:

$$\varepsilon_t = X_t + \sum_{j=1}^{\infty} \mathbf{1}' \left[\prod_{i=0}^{j-1} (\Psi_1 I_{1,t-d-i} + \Psi_2 I_{2,t-d-i} + \dots + \Psi_k I_{k,t-d-i}) \right] \mathbf{X}_{t-j}. \quad (12)$$

Proof. The proof follows the same steps illustrated in Theorem 1. $\square$

3 Empirical results

To empirically illustrate the theoretical results presented until now, consider a TMA(2;2) model:

$$X_t = \varepsilon_t - (\theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2}) I_{t-d} - (\phi_1 \varepsilon_{t-1} + \phi_2 \varepsilon_{t-2})(1 - I_{t-d}), \quad (13)$$

with $\{\varepsilon_t\}$ a sequence of independent Gaussian random variables $N(0, 1)$. When model (13) is invertible, it follows that:

$$\varepsilon_t = X_t + \sum_{j=1}^{\infty} \pi_{j,t} X_{t-j}, \quad (14)$$

which can be approximated with

$$\varepsilon_{nt} = X_t + \sum_{j=1}^n \pi_{j,t} X_{t-j}, \quad (15)$$

for n adequately large, where $\lim_{n \rightarrow \infty} E[(\varepsilon_t - \varepsilon_{nt})^2] = 0$ and the coefficients $\pi_{j,t}$ are:

- $\pi_{1,t} = \theta_1 I_{t-d} + \phi_1 (1 - I_{t-d})$,
- $\pi_{2,t} = \pi_{1,t} [\theta_1 I_{t-d-1} + \phi_1 (1 - I_{t-d-1})] + \theta_2 I_{t-d} + \phi_2 (1 - I_{t-d})$,
- $\pi_{j,t} = \pi_{j-1,t} [\theta_1 I_{t-d-(j-1)} + \phi_1 (1 - I_{t-d-(j-1)})] + \pi_{j-2,t} [\theta_2 I_{t-d-(j-2)} + \phi_2 (1 - I_{t-d-(j-2)})]$, for $j = 3, 4, \dots, n$.

Taking advantage of the results in Theorem 1, it is interesting to note that the convergence of the sum in model (15) is guaranteed not only when both regimes are locally invertible but even in presence of regimes that are locally noninvertible. To better discuss this point consider the following example.

Example 1. Let X_t a TMA(2;2) process and generate $T = 10000$ observations (with burn-in 1000) with $d = 1$, null threshold value and with parameters $\theta_1 = 1.7, \theta_2 = -0.7$ in the first regime and $\phi_1 = 1.2, \phi_2 = -0.6$ in the second regime.

The matrix Θ has now eigenvalues $\lambda_1 = 1, \lambda_2 = 0.7$, whereas the matrix Φ has eigenvalues $\lambda_{1/2} = 0.6 \pm 0.4899i$.

The eigenvalues of both matrices highlight that the first regime is locally noninvertible but, as shown in Fig. 1 (left), the convergence of the sum in model (15) is always reached exponentially. $\square$

The results in Example 1 have remarkable interest: first we can state that the TMA model can be invertible even in presence of noninvertible regimes; second the definition of the $\pi_{j,t}$ parameters allows to extend, with the proper complications, some result widely known for the moving average models of order $q = 2$ (see [1]); third, when $I_{t-d} = 0$ or $I_{t-d} = 1$ the results presented match those given in the context of the MA(2) model.

Note that the convergence to zero, as $n \rightarrow \infty$, of the sum in model (15) can be differently reached as shown in the following example.

Example 2. Let $X_t \sim TMA(2;2)$ with parameters $\theta_1 = 0, \theta_2 = 1, \phi_1 = -0.4, \phi_2 = 0.3, d = 1$ and threshold value $r = 0$. The eigenvalues of Θ are $\lambda_{1/2} = \pm 1$ and the eigenvalues of Φ are $\lambda_1 = -0.7831$ and $\lambda_2 = 0.3831$. In this case, the simulated process, of length $T = 10000$, is still globally invertible (even if the first regime has two unit roots) and the sum in (15) converges to zero as shown in Fig. 1 (middle). $\square$

Example 3. To further investigate the difference between the local and the global invertibility and to further discuss the theoretical results given in the previous pages, consider the following TMA(2;2) process:

$$X_t = \varepsilon_t - (0.4\varepsilon_{t-1} + 0.6\varepsilon_{t-2})I_{t-1} - (-0.5\varepsilon_{t-1} + 0.5\varepsilon_{t-2})(1 - I_{t-1}) \quad (16)$$

with threshold value $r = 0$. The matrices Θ and Φ associated to the process (16) have eigenvalues respectively $\lambda_{1,\Theta} = 1, \lambda_{2,\Theta} = -0.6$ and $\lambda_{1,\Phi} = -1, \lambda_{2,\Phi} = 0.5$. The parameters used in the simulation imply that the process under analysis is locally noninvertible in both regimes but when we evaluate the convergence of the coefficients $\pi_{j,t}$ in Fig. 1 (right), we note that the global invertibility is preserved. $\square$

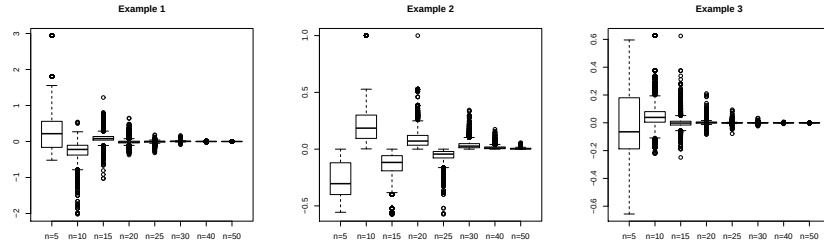


Fig. 1 Box-plots of the $\pi_{j,t}$ parameters of model (15), for $n = 5, 10, 15, 20, 25, 30, 40, 50$, obtained from Example 1 (left), Example 2 (middle) and Example 3 (right)

The main consequence of the last example is that the invertibility of the TMA(2;2) model can be obtained even when $|\lambda(\Theta)|^p |\lambda(\Phi)|^{1-p} = 1$. It should allow to weaken the inequality in Theorem 1 that will be object of further research.

Appendix

In the proof on Theorem 1 the following notation is used.

Given the sequence of $(q \times q)$ matrices $\mathbf{A}_n$ and let $\lambda_i(\mathbf{A}_n)$ the i -th eigenvalue of $\mathbf{A}_n$ (for $i = 1, 2, \dots, q$), then we say that the sequence $\mathbf{B}_n$ is dominant with respect to $\mathbf{A}_n$ ($\mathbf{A}_n \prec \mathbf{B}_n$), if the dominant eigenvalue (the maximum absolute eigenvalue) of $\mathbf{B}_n$, denoted $\lambda(\mathbf{B}_n)$, is greater or equal to the dominant eigenvalue of $\mathbf{A}_n$, $\lambda(\mathbf{A}_n)$, for $n = 1, 2, \dots$.

In other words:

$$\mathbf{A}_n \prec \mathbf{B}_n, \quad \text{if } \lambda(\mathbf{A}_n) \leq \lambda(\mathbf{B}_n) \quad (17)$$

From (17) it follows that:

1. if $\lambda(\mathbf{B}_n) \rightarrow 0$, as $n \rightarrow \infty$, then $\mathbf{B}_n \rightarrow \mathbf{0}_{(q \times q)}$ and $\mathbf{A}_n \rightarrow \mathbf{0}_{(q \times q)}$, with $\mathbf{0}$ a matrix of zeros;
2. given the sequence $\mathbf{A}_n$, then $\mathbf{A}_n \prec \lambda(\mathbf{A}_n)\mathbf{I}$, where $\mathbf{I}$ is the identity matrix of order q .

References

1. Box, G.E.P., Jenkins, G.M.: Time series analysis, forecasting and control. Holden-Day, San Francisco (1976).
2. De Gooijer, J.G.: On threshold moving-average models, J. Time Ser. Anal. **19**, 1-18 (1998)
3. De Gooijer, J.G., Brännäs, K.: Invertibility of non-linear time series models, Commun. Stat.: Theory and Methods, **24**, 2701-2714 (1995)
4. Li, G., Li, W.K.: Testing for threshold moving average with conditional heteroscedasticity, Stat. Sin., **18**, 647-665 (2008)
5. Ling, S., Tong, H.: Testing for a linear MA model against threshold MA models, Ann. Stat. **33**, 2529-2552, (2005)
6. Ling, S., Tong H., Li, D.: Ergodicity and invertibility of threshold moving-average models, Bernoulli, **13**, 161-168 (2007)
7. Tong, H.: Non-linear time series. A dynamical system approach. Clarendon Press Oxford (1990)