

Testing Determinacy Versus Indeterminacy and Omitted Dynamics in Linear Rational Expectations Models

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Abstract Testing the null hypothesis that a multivariate linear rational expectations (LRE) model admits a unique stable solution (determinacy) against the alternative of multiple stable solutions (indeterminacy) involves two major difficulties: (i) a set of nuisance parameters that index the VARMA-type indeterminate reduced forms are not identifiable under the null of determinacy; (ii) indeterminacy can be confused with the omission of relevant lags/leads in the LRE model. We address the points (i) and (ii) by proposing a simple testing strategy which exploits the properties of both full- and limited-information estimation techniques and combines two standard tests. First, for suitable choices of instruments and weighting matrix, the LRE model is estimated through generalized instrumental variables (GIV) techniques and the J-test computed. If the J-test does not reject the LRE model, the VAR-type determinate solution is estimated through maximum likelihood (ML) and the implied set of cross-equation restrictions (CER) are tested by a Lagrange multiplier (LM) or likelihood ratio (LR) test. If the LM (LR) test does not reject the CER the hypothesis of determinacy is accepted otherwise indeterminacy is selected. The proposed testing strategy entails a multiple hypothesis testing issue which can be easily addressed.

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1 Introduction²

It is well known that linear rational expectations (LRE) models can have multiple equilibria, a situation referred as indeterminacy. A determinate LRE model has a unique stable (asymptotically stationary) reduced form solution. The time series representation of a determinate LRE model may differ substantially from that of an indeterminate LRE model, and, in addition, non-fundamental stochastic disturbances, which do not appear in the former, can influence the dynamics of the latter. The effect of indeterminacy is to admit additional possible sources of fluctuations to an economic system: in addition to shocks to preferences and technologies, the dynamics of the system may also be driven by the self-fulfilling beliefs of economic agents. While many economists interpret this phenomenon as a reflection of reality and a possible explanation of a wide range of economic phenomena, others argue that indeterminacy signals coordination problems and affects what could be called the “predictability” of the system. Various selection criteria have been suggested for choosing one among the many possible equilibria, see Driskill (2006) for a recent review.

Recently, macroeconomists have become interested in evaluating the determinacy/indeterminacy of a particular class of linear(ized) LRE models belonging to the New-Keynesian tradition, known as small-scale dynamic stochastic general equilibrium (DSGE) monetary models. The equilibrium implied by these models may not be unique if the central bank does not raise sufficiently the nominal interest rate in response to inflation, see e.g. Lubik and Schorfheide (2004) and Benati and Surico (2009). The phenomenon known in the literature as “the Great moderation”, has been extensively investigated and explained in terms of the indeterminacy/determinacy of the LRE model used to explain the economy.

On the econometric side, the non-uniqueness problem in the econometrics of LRE models has been traditionally associated with a particular type of “explosive indeterminacy” that may arise in models for financial asset markets and foreign exchange markets, or in the Cagan's monetary model of hyperinflation, usually called rational bubbles, see, *inter alia*, Hamilton and Whiteman, (1985). Aside from rational bubbles, however, only a few of studies focus on the econometrics of determinacy/indeterminacy in stable LRE models. To our knowledge, Lubik and Schorfheide (2004) is the only existing contribution which addresses the issue of

² Many technical aspects and results summarized in this article are discussed and treated extensively in a companion version of the paper, see Fanelli (2010).

testing determinacy/indeterminacy in LRE models. Lubik and Schorfheide (2004) adopt a Bayesian perspective; this article attempts to extend the analysis to a classical framework.

This paper is organized as follows. Section 2 introduces the LRE model, its reduced form solutions and discusses the main shortcomings involved with the econometric analysis of determinacy/indeterminacy. Section 3 summarizes the suggested testig procedure and Section 4 reports some Monte Carlo experiments which shed light on the size of the procedure.

2 Model

We consider a family of multivariate asymptotically stable LRE models which covers many of the models currently used in finance and macroeconomics. The system is given by

$$(1) \quad \begin{aligned} \Gamma_0 X(t) &= \Gamma_f E_t X(t+1) + \Gamma_b X(t-1) + \omega(t) \\ \omega(t) &= R \omega(t-1) + u(t) \end{aligned}$$

where $X(t)$ is the $p \times 1$ vector of observable variables, $E_t \bullet := E(\bullet | \Omega(t))$ is the expectations operator conditional upon the information set available at time t , $\Omega(t)$, Γ_a , $a=0,f,b$ are $p \times p$ matrices containing the structural parameters of the LRE model (Γ_0 is non-singular), $\omega(t)$ is a $p \times 1$ structural disturbance, R is a $p \times p$ diagonal (stable) matrix, and $u(t)$ is a $p \times 1$ martingale difference sequence with respect to $\Omega(t)$ with finite covariance matrix.

Let γ be vector of structural parameters, i.e. the free elements in Γ_a , $a=0,f,b$ and R . In the class of LRE models in Eq. (1), if a unique stable solution exists, it can be represented as the VAR-type process

$$(2) \quad \begin{aligned} X(t) &= H Y(t) \\ Y(t) &= \Phi(\gamma) Y(t-1) + \Psi(\gamma) U(t) \end{aligned}$$

where $Y(t) = (X(t)', X(t-1)')$ is the $2p \times 1$ state vector, $U(t) = (u(t)', 0')$, H is a known $p \times 2p$ selection matrix, $\Psi(\gamma)$ is a $2p \times 2p$ non-singular matrix which depends nonlinearly on γ ,

$\Phi(\gamma)$ is a $2p \times 2p$ stable matrix (i.e. with all of its eigenvalues inside the unit disk) which depends nonlinearly on γ by the quadratic matrix equation

$$(3) \quad \Gamma_f^c [\Phi(\gamma)]^2 - \Gamma_0^c \Phi(\gamma) + \Gamma_c^c = 0$$

where Γ_f^c , Γ_0^c and Γ_c^c are the companion form counterparts of the matrices Γ_a , $a=0,f,b$, respectively, see Fanelli (2010). Eq. (3) defines the implicit set of CER that the LRE model places on its reduced form solution. It can be shown that the solution in Eq. (2) is (locally) unique if the $p \times p$ matrix $S(\gamma) = (\Gamma_0 + R\Gamma_f - \Gamma_f \Phi_1(\gamma))^{-1} \Gamma_f$ is stable (i.e. with eigenvalues inside the unit circle), where $\Phi_1(\gamma)$ is the top-upper-left $p \times p$ block of $\Phi(\gamma)$.

Conversely, if multiple stable solutions occur, the matrix $S(\gamma)$ has p_2 ($0 < p_2 \leq p$) eigenvalues that lie outside (or on) the unit circle, leading to a substantial violation of the structure in Eq. (2). A detailed derivation of the implied reduced forms can be found in Fanelli (2010). In summary, the indeterminate solutions can be described by the vector autoregressive moving average (VARMA)-type processes

$$(4) \quad \begin{aligned} X(t) &= H Y(t) \\ Y(t) &= \Pi_1(\gamma) Y(t-1) + \Pi_2(\gamma) Y(t-2) + M_1(k_0, \gamma) U(t) + M_2(\gamma) U(t-1) \end{aligned}$$

where the $2p \times 2p$ matrices $\Pi_1(\gamma)$ and $\Pi_2(\gamma)$ and $M_2(\gamma)$ depend nonlinearly on γ , while the $2p \times 2p$ matrix $M_1(k_0, \gamma)$ also depends on a set of nuisance (auxiliary) parameters contained in the $(p_2) \times 1$ vector k_0 , which make the VMA part of the solution arbitrary. The parameters do not depend on the structural parameters and are not identifiable under the null of determinacy.

Actually, the VARMA-type solutions in Eq. (4) can be characterized by an additional source of indeterminacy, given by the presence of unobservable “extraneous” stationary stochastic processes, called sunspots, enter the right-hand-side of Eq. (4) complicating the dynamics and volatility of the system. Sunspot shocks are usually ascribed to agents’ self-fulfilling beliefs.

Two shortcomings complicate the analysis.

2.1 Shortcoming 1

If the testing problem is addressed by comparing the likelihood functions of the determinate (Eq. (2)) and indeterminate reduced forms (Eq. (4)), the typical context is that in which a set of nuisance parameters which enters the alternative model is not identifiable under the null. In other words, one faces the problem of comparing the likelihood of the highly constrained VARMA(2,0) process in Eq. (2), with the likelihood of the highly constrained VARMA(2+1,0+1) process in Eq. (4). As is known, these testing problems involve standard distributions only if the parameters which are unidentifiable under the null are known; when these parameters are unknown, the asymptotic theory becomes non-standard and involves the so-called “cancelling roots” phenomenon. The specific case in which an univariate ARMA(0,0) process is tested against an ARMA(1,1) process is covered by Hannan (1982) and Andrews and Ploberger (1996) and is generalized to higher-order univariate ARMA models in Veres (1987). Unfortunately, the general solution, discussed in Andrews and Ploberger (1994) is not easily adaptable in the multivariate framework because of the difficulties in envisaging all the directions of the parameter space of the LRE model that lack of identification (Lubik and Schorfheide, 2004).

2.2 Shortcoming 2

Since the VARMA-type solutions in Eq. (4) have a richer dynamic structure compared to the VAR solutions obtained under determinacy, one can erroneously confound the possible omission of relevant propagation mechanisms in the LRE model (i.e. additional lags and/or leads) with indeterminacy. This means that if the researcher firmly believes in the validity of the LRE model and treats it as the data generating process, the indeterminate solution will be selected also when the true data generating process is represented by the determinate solution of a counterpart of the LRE model involving more lags and/or leads than specified by the econometrician. According to Lubik and Schorfheide (2004), all system-based approaches to evaluation of indeterminacies are affected by this weakness.

3 Testing strategy

We address the two above mentioned shortcomings through a new, computationally simple, testing procedure. The procedure is based on the estimation of the LRE model by both “full-” and “limited-information” techniques, and relies on the sequential use of two standard tests.

As is known, the main advantage of limited-information methods in LRE models based on valid instruments is that estimation is directly based on the structural (Euler) equations comprising the system so that estimation is robust to indeterminacy/indeterminacy. On the other hand, full-information techniques provide more efficient estimates if the researcher specifies the correct reduced form. Thus, if for instance the solution is indeterminate (VARMA-type) but the researcher specifies a determinate reduced form (VAR-type), the estimator of the structural parameters retrieved from the CER will be inconsistent.

These considerations suggest a simple testing strategy which circumvents the inferential and computational complications involved with the estimation of the reduced forms in Eq. (4). More importantly, it accounts for the possible (dynamic) misspecification of the LRE model. The procedure works as follows. Estimate the parameters γ of the LRE model in Eq. (1) by Generalized Instrumental Variables (GIV), using the (relevant) vector of over-identifying instruments $Z(t) := (X(t-1)', X(t-2)')$ irrespective of whether the solution of the LRE model is unique or multiple, and consider a parametric estimator of the weighting matrix which accounts for the VMA structure of the disturbances as in Cumby *et al.* (1983) or West (1997). Estimate the parameters γ of the LRE model in Eq. (1) by maximum likelihood (ML), confining attention to the determinate reduced form in Eq. (2) subject to the CER in Eq. (3); the constraints that ensure that the estimated matrix $S(\gamma)$ has eigenvalues inside the unit circle are explicitly imposed in estimation.³ Then consider the following sequence of standard tests:

Test 1. Compute the over-identification restrictions test (J-test). If the J-test rejects, rejects the LRE model. If the J-test does not reject compute Test 2 below.

Test 2. Compute a Lagrange multiplier (LM) (or likelihood-ratio, LR) test for the CER in Eq. (2). If the LM (LR) does not reject the CER accept determinacy, otherwise select indeterminacy.

³ See Fanelli (2010), Appendix B, for a detailed discussion of the issue.

The procedure based on the sequence Test 1–Test 2 involves a multiple hypotheses testing issue which can be easily addressed. To see this, let $E1$ be the event: “the J-test rejects the LRE model ” and $E2$ the event: “the LM (LR) test rejects the CER”; $E1^c$ is the complementary event of $E1$. One has

$$(5) \quad \Pr[\text{Test 1-Test 2 rejects}] = \Pr[(E2 \cap E1^c)] / (1 - \Pr[E1]) + \Pr[E1]$$

hence the evaluation of the size of the procedure and the power against the alternatives of indeterminacy and omitted dynamics is relatively easy if $\Pr[(E2 \cap E1^c)]$ is evaluated through Monte Carlo techniques.

4 Some Monte Carlo evidence

Table 1 investigates the small-sample size ($T=70$, $T=150$) properties of the testing procedure considering a data generating process which is described in detail in Table 2 of Fanelli (2010). In short, $M=1000$ samples of length T have been generated from a determinate LRE model, see Eq.s (2)-(3), where the structural parameters γ have been chosen such that they are close to the bound which separates the determinacy from the indeterminacy region of the parameter space. For each generated sample, the J-test has been computed from the direct GIV estimation of Eq. (1) by using the relevant vector of instruments $G(t) := (X(t-1)', X(t-2)')'$ and a parametric estimator of the weighting matrix which accounts for the VMA structure disturbances; likewise, the LM test for the CER has been computed by ML from the VAR in Eq. (2) subject to Eq. (3). Table 1 summarizes the empirical rejection frequencies (i.e. the number of times out of M in which the test rejects the null hypothesis) of Test 1 and Test 2 considered isolation and then jointly (Test 1-Test 2), considering a 5% Type I error; the latter is calculated by using Eq. (5).

Table 1: Empirical rejection frequencies under determinacy (5% nominal size)

<i>DGP:</i> Table 2 in Fanelli (2010)		
	$T=150$	$T=70$
Test 1: J-test for LRE model	0.064	0.064
Test 2: LM test for CER	0.021	0.022
Overall: Test 1 – Test 2	0.041	0.066

The results in Table 1 show that for samples of length typically encountered in practice, and for values of γ close to the boundaries of the indeterminacy region of the parameter space, the sequence Test 1-Test 2 provides a substantial control of the null hypothesis. The investigation of the power against indeterminacy and omitted dynamics is the subject of an ongoing research project.

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