

# Portfolio optimization in defaultable markets under incomplete information

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**Key words:** Portfolio optimization, partial information, credit risk, dynamic programming, robust solutions.

## 1 Introduction

Our study concerns the classical portfolio optimization problem of maximization of expected utility of terminal wealth when the assets, in which one invests, may default. We put ourselves in a context, where the dynamics of the asset prices are affected by exogenous factor processes, some of which may have an economic interpretation, some may not but, most importantly, not all of them may be directly observable. In credit risk models factors are often used to describe contagion: physical and information induced. Information induced contagion arises due to the fact that the successive updating of the distribution of the latent (not observable) factors in reaction to incoming default observations leads to jumps in the default intensity of the surviving firms (this is sometimes referred to as “frailty approach”, see e.g. [6]). As shown in [4], unobservable factor processes are needed on top of observable covariates in order to explain clustering of defaults in historical credit risk data. In general, the formulation of a model under incomplete information on the factors allows for greater model flexibility, avoids a possible inadequate specification of the model itself, and the successive updating of the distribution of the unobserved

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factors (for constant factors one considers them from the Bayesian point of view as random variables) allows the model to “track the market” thus avoiding classical model calibration.

To keep the presentation at a possibly simple level, we shall consider only a single factor process that is supposed not to be directly observable and the observation history is given, in addition to the defaults, by the observed asset prices. Furthermore, we shall consider discrete time dynamics. With respect to continuous time models, this can be justified since trading takes place in discrete time anyway. Moreover, a solution is easier to compute in discrete time and, while it is more difficult to obtain qualitative results than in continuous time, once an explicit numerical solution is obtained, one can evaluate its performance also with respect to alternative criteria via simulation.

## 2 The model

Here we describe the model dynamics and the objective for our portfolio optimization.

### 2.1 Model dynamics

Given a discrete time set  $t_0 = 0 < t_1 < \dots < t_N = T$ , a filtered probability space  $(\Omega, \mathcal{G}, \mathcal{G}_t, P)$  ( $\mathcal{G}$  stands for “global filtration”) and, in addition to a nonrisky asset with price  $S_n^0$  ( $S_n^0$  is the price at time  $t_n$ ), a set of  $M$  risky assets with prices  $S_n^m$ ,  $m = 1, \dots, M$  that are subject to default except for the first one  $S_n^1$  (both for applications as well as for formal reasons it is convenient to consider investment in at least one default-free asset). Let  $\tau^m$  be the exogenously given default time of the  $m$ -th asset and consider the default indicator process

$$H_n := (H_n^1, \dots, H_n^M), \quad n = 0, \dots, N \quad (1)$$

where  $H_n^m := \mathbf{1}_{\{t_n \geq \tau^m\}}$  is the default indicator process for the  $m$ -th firm. The possible values of  $H_n$  are the  $M$ -tuples  $h^p = (h^{p,1}, \dots, h^{p,M})$  for  $p = 1, \dots, 2^M - 1$  with  $h^{p,m} \in \{0, 1\}$ . Since  $S_n^1$  is assumed to be default free, we have  $H_n^1 \equiv 0$  and an ordering of the values  $h^p$  is supposed to be given whereby, typically,  $h^1 = (0, 0, \dots, 0)$  and  $h^{2^M - 1} = (0, 1, \dots, 1)$ .

We let now the dynamics of the asset prices be given by

$$\begin{cases} S_{n+1}^0 = S_n^0(1 + r_n) & (\text{typically } r_n \equiv r) \\ S_{n+1}^m = S_n^m \gamma^m(\xi_{n+1})(1 - H_{n+1}^m); \quad m = 1, \dots, M \end{cases} \quad (2)$$

where  $\xi_n$  is a sequence of iid multinomial random variables with values  $\xi_n \in \{\xi^1, \dots, \xi^L\}$ . In vector form we may then write

$$S_{n+1} = \text{diag}(S_n^m \gamma^m(\xi_{n+1})) (1 - H_{n+1}) := I(S_n, \xi_{n+1}, H_{n+1}) \quad (3)$$

The price evolution is thus driven by  $(\xi_n, H_n)$ , defined on  $(\Omega, \mathcal{G}, \mathcal{G}_n, P)$  as follows. Given an  $\mathcal{G}_n$ -adapted finite state Markov chain  $Z_n$  with values  $Z_n \in \{z^1, \dots, z^J\}$  and transition probability matrix

$$P^{ij} := P\{Z_n = z^j \mid Z_{n-1} = z^i\}; \quad \forall i, j \in \{1, \dots, J\}, \quad \forall n \text{ (time homogeneous)}, \quad (4)$$

the driving processes  $(\xi_n, H_n)$  are supposed to be independent, conditionally on  $Z_n$ , with distribution characterized by assigning

$$\begin{cases} p^\ell(z) &:= P\{\xi_n = \xi^\ell \mid Z_n = z\}, \quad \ell = 1, \dots, L; \quad \forall n \leq N \\ \rho^{p,q}(z) &:= P\{H_n = h^q \mid H_{n-1} = h^p, Z_n = z\} \\ &\quad \forall p, q \in \{1, \dots, 2^{M-1}\} \quad \forall n \leq N \end{cases} \quad (5)$$

Notice that the dependence of  $\rho^{p,q}$  on  $\{Z_n\}$  allows to model contagion: only physical if  $Z_n$  is observed and “information-induced” if  $Z_n$  is unobserved and its distribution updated on the basis of the observed defaults and defaultable asset prices.

## 2.2 Portfolios

To perform portfolio optimization we need to invest in the market and for this purpose we consider an investment strategy that may be defined either by specifying the number of units invested in the individual assets, namely  $a_n = (a_n^0, a_n^1, \dots, a_n^M)$  ( $a_n^m$  is the number of units of asset  $m$  held in the portfolio in period  $t_n$ ), or by equivalently specifying the ratios invested in the individual assets, i.e.  $\phi_n^0 = \frac{a_{n+1}^0 S_n^0}{V_n^\phi}$ ,  $\phi_n^m (1 - H_n^m) = \frac{a_{n+1}^m S_n^m}{V_n^\phi}$  where  $V_n^\phi = V_n^a := \sum_{m=0}^M a_n^m S_n^m$  is the portfolio value in period  $t_n$ ; notice also that  $\phi_n^0 = 1 - \sum_{m=1}^M \phi_n^m (1 - H_n^m)$  so that, to define a self financing investment strategy  $\tilde{\phi}_n := (\phi_n^0, \phi_n^1, \dots, \phi_n^M)$ , it suffices to define  $\phi_n := (\phi_n^1, \dots, \phi_n^M)$ . With the given definitions one has that investment in an asset automatically ceases as soon as it defaults (implies the equivalence of the expressions for  $V_n^a$  and  $V_n^\phi$ ).

Assuming first that the factor process  $Z_n$  is observed by an investor, the definitions above also imply that we consider  $a_n$  to be a predictable process ( $a_n$  is  $\mathcal{G}_{n-1}$ -measurable), while  $\phi_n$  are adapted.

### 2.3 The partial information problem

In view of formulating our partial information problem, let the default history be given by the filtration  $\mathcal{H}_n := \sigma\{H_v, v \leq n\}$ . With this filtration we can reexpress the global filtration as  $\mathcal{G}_n = \mathcal{F}_n^S \vee \mathcal{H}_n \vee \mathcal{F}_n^Z$  while, with  $\mathcal{F}_n^S$  denoting the filtration given by the price observation history, the observation filtration (representing the information of an investor) is given by  $\mathcal{F}_n = \mathcal{F}_n^S \vee \mathcal{H}_n \subset \mathcal{G}_n$ .

Having specified an utility function  $u(\cdot)$  satisfying the usual conditions, we can now give the following

**Definition 1** *A self financing investment strategy  $\phi_n$  is called admissible in our partial information problem if, besides implicit technical conditions, it is  $\mathcal{F}_n$ -adapted and such that  $V_n^\phi$  belongs to the domain of  $u(\cdot)$ .*

Notice that, in general, the set of admissible strategies is non empty (in the log and power utility cases it contains the strategy of not investing in the risky assets) and is a convex set that may be unbounded; by possibly bounding the domain (e.g.  $\phi_n^m \geq -C$ ) it can be made compact. We come now to define our

**Problem:** Determine an admissible  $\{\phi^*\}$  such that

$$E \left\{ u \left( V_N^{\phi^*} \right) \right\} \geq E \left\{ u \left( V_N^\phi \right) \right\}, \quad \forall \text{ admissible } \phi$$

Our problem is a partial information problem in that the factor process  $Z_n$  cannot be observed; on the other hand the investment strategy can depend only on observable quantities. The usual approach in this situation is (see e.g. [1], [2], [7], see also [3] for a problem related to the one of the present paper) to transform the partial observation problem into one of full information by replacing the unobservable quantities  $Z_n$  by their conditional distributions, given the current observation history. These conditional distributions are called *filter distributions* or just *filter* and they can be computed recursively as we are going to show in the next section.

### 3 Filter

Since the investment strategy  $\phi_n$  is  $\mathcal{F}_n$ -adapted, the information coming from observing  $(S, V, H)$  (namely the asset prices, the portfolio value, and the defaults) is equivalent to that of observing just  $(S, H)$ .

Putting  $(S^n, H^n) := ((S_1, H_1), \dots, (S_n, H_n))$ , the filter distribution for  $Z_n$  is  $\pi_n = (\pi_n^1, \dots, \pi_n^J)$  with

$$\pi_n^j := P \{ Z_n = z^j \mid (S^n, H^n) \}, \quad j = 1, \dots, J; \quad \forall n \leq N$$

By applying the *recursive Bayes' formula* one obtains, for  $j = 1, \dots, J$ ,

$$\begin{aligned}
\pi_n^j &= P \{ Z_n = z^j \mid (S_n = s_n, H_n = h_n), (S^{n-1}, H^{n-1}) \} \propto \\
&\propto P \{ (S_n = s_n, H_n = h_n) \mid Z_n = z^j, S_{n-1}, H_{n-1} \} \sum_{i=1}^J P^{ij} \pi_{n-1}^i
\end{aligned} \tag{6}$$

with the observation distribution (likelihood function) given by

$$\begin{aligned}
P \{ (S_n = s_n, H_n = h^q) \mid Z_n = z^j, S_{n-1}, H_{n-1} = h^p \} = \\
\rho^{p,q}(z^j) \sum_{\ell=1}^L p_\ell(z^j) \mathbf{1}_{\{s_n = I(s_{n-1}, \xi^\ell, h^q)\}} := F(z^j; s_n, s_{n-1}, h^q, h^p)
\end{aligned} \tag{7}$$

where  $I(s, \xi, h)$  was defined in (3). Notice that, since the model may not correspond exactly to reality, there may be no  $\xi^\ell \in \{\xi^1, \dots, \xi^L\}$  so that, for the actually observed  $s_{n-1}, s_n$ , one has  $s_n = I(s_{n-1}, \xi^\ell, h^q)$ . This latter equality has thus to be understood in the “nearest neighbor” sense.

Putting  $F(s_n, s_{n-1}, h_n, h_{n-1}) := \text{diag} (F(z^j; s_n, s_{n-1}, h_n, h_{n-1}))$ , which is an  $J \times J$ -matrix, the recursions (6) can be expressed in vector form as

$$\pi_n = \frac{F(s_n, s_{n-1}, h_n, h_{n-1}) P \pi_{n-1}}{|F(s_n, s_{n-1}, h_n, h_{n-1}) P \pi_{n-1}|_1} := \bar{F}(\pi_{n-1}, s_n, s_{n-1}, h_n, h_{n-1}) \tag{8}$$

#### 4 Dynamic Programming for the equivalent full information problem

Under full information corresponding to  $\mathcal{G}_n$  the tuple  $(S, V, H, Z)$  is Markov. In the full information equivalent of the partial information problem the process  $Z_n$  has to be replaced by the filter process  $\pi_n$  and from (6) it is easily seen that, in the filtration  $\mathcal{F}_n$ , it is the tuple  $(S, V, H, \pi)$  that is Markov.

Denoting by  $U_n(s, v, h, \pi)$  the optimal value in period  $t_n$  for  $S_n = s, V_n = v, H_n = h, \pi_n = \pi$ , i.e.

$$U_n(s, v, h, \pi) = \sup_{\phi \text{ adm.}} E \{ u(V_N^\phi) \mid S_n = s, V_n = v, H_n = h, \pi_n = \pi \}$$

the principle of Dynamic Programming (see e.g. [2]) leads to the backwards recursions

$$\begin{cases} U_N(s, v, h, \pi) = u(v) & \text{and, for } n \leq N \\ U_{n-1}(s, v, h, \pi) = \\ \max_{\phi_{n-1}} E \{ U_n(S_n, V_n, H_n, \pi_n) \mid (S, V, H, \pi)_{n-1} = (s, v, h, \pi) \} \end{cases} \tag{9}$$

### 4.1 Explicit solution in the log-utility case

In the log-utility case (analogously in the power utility case) we have (see Chapter 2 of [5])

**Theorem 2** For  $n = 1, \dots, N$  and assuming  $H_n = h^p$  for some  $p \in \{1, \dots, 2^{M-1}\}$  we have

$$U_n(s, v, h^p, \pi) = \log v + K_n(s, h^p, \pi)$$

with

$$\begin{cases} K_N(s, h^p, \pi) = 0 \\ K_n(s, h^p, \pi) = k(h^p, \pi) + \sum_{i=1}^J \pi^i \sum_{\ell=1}^L p_\ell(z^i) \sum_{q=1}^{2^{M-1}} \rho^{p,q}(z^i) \cdot \\ \cdot K_{n+1}(I(s, \xi^\ell, h^q), h^q, \bar{F}(\pi, I(s, \xi^\ell, h^q), s, h^q, h^p)) \end{cases}$$

where  $\bar{F}(\cdot)$  was defined in (8) and where

$$\begin{aligned} k(h^p, \pi) &= \max_{\phi=(\phi^1, \dots, \phi^M)} \left\{ \sum_{i=1}^J \pi^i \sum_{\ell=1}^L p_\ell(z^i) \sum_{q=1}^{2^{M-1}} \rho^{p,q}(z^i) \cdot \right. \\ &\quad \left. \cdot \log \left[ (1+r) + \sum_{m=1}^M \phi^m (1-h^{p,m}) \left[ \gamma^m(\xi^\ell) (1-h^{q,m}) - (1+r) \right] \right] \right\} \end{aligned}$$

Notice that, in each period  $n$ , the additive term  $K_n(\cdot)$  results thus from the sum of a current additive term  $k(h^p, \pi)$  and the conditional expectation of the previous  $K_{n+1}(\cdot)$ . The proof can rather straightforwardly be obtained by backwards induction on  $n$ .

### 4.2 Particular case of $Z_n \equiv Z$ unobserved

In the case of  $Z_n \equiv Z$  the factor process reduces to an unobserved parameter that, in accordance with the Bayesian point of view, is considered as a random variable  $Z$ . Even if  $Z$  is modeled as not time varying, the successive updating of its distribution, i.e.

$$\pi_n^j := P\{Z = z^j \mid (S^n, H^n)\}, \quad j = 1, \dots, J, \quad n \leq N$$

makes the context dynamic; the solution is obtained as in the general case and here it simplifies considerably. In fact, in the filter dynamics (6) the recursive Bayes' formula reduces to the ordinary one that here becomes

$$\pi_n^j \propto P\{(S_n = s_n, H_n = h_n) \mid Z = z^j, S_{n-1}, H_{n-1}\} \cdot \pi_{n-1}^j$$

## 5 Comments on numerical aspects and the issue of robustness

Numerical calculations for the case of  $Z_n \equiv Z$  unobserved are reported in Chapter 2 of the thesis [5]. The solution is computed there for various values of the model parameters in the log-utility case when  $M = 3$  (one non defaultable and two defaultable risky assets which is the smallest number to have the possibility of contagion),  $L = 2$  (asset price evolution according to a binomial model), and  $Z$  taking just two possible values/regimes:  $Z = 1$  “bad state” (bear market),  $Z = 0$  “good state” (bull market).

The computations were done having two goals in mind: i) investigating the effect of allowing for shorting in the risky assets; ii) investigating the “robustness” of the solution. Among the findings for point i) it turned in particular out that, in the case of no shorting, in the bad state one does in general not invest in the risky assets. When shorting is allowed then, up to a certain level of “return” on the risky assets, the optimal value in the bad state is superior to that in the good state, which is due to the fact that the returns on the defaultable assets make it convenient to go short in them; beyond that level the optimal value in the good state is superior than in the bad state as one would expect.

For what concerns point ii) the *robustness* is in the sense of obtaining a solution that works well for a variety of possible models. This is an important issue because the “exact model” is practically never known and, on the other hand, the solution may be rather sensitive to the model. From the numerical calculations it turned out rather clearly that the solution obtained for the model under incomplete information possesses this property of robustness in the sense that, while it underperforms the one under a hypothetical full information on the model, it performs much better with respect to using the wrong solution for the wrong model.

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