

Modelling the Dependence in Multivariate Extremes

Matteo Grigoletto

Abstract In the analysis of multidimensional extreme observations we are often interested in understanding the dependence structure of the marginals, in order to define measures to be used, for example, in risk management. With this aim, many models have been built using copula functions developed both with a nonparametric approach and with families of parametric models. A limit in the use of copula functions in this context is due to the fact that there isn't a finite parametric model which can express all the possible shapes of the multivariate extreme-value copula. In this paper we propose a different model to estimate the dependence structure of extreme values, based on the analysis of independent components, with sources distributed as the generalised extreme-value random variable. Differently from copula analysis, this approach does not require knowledge of the multivariate extreme-value distribution which could have generated the sample data. An application to the estimation of the value-at-risk for extreme returns relating to a portfolio of stock indices and to a portfolio of assets shows the utility of the proposed methodology.

Key words: Statistics of extremes, Copula, Independent Component Analysis, Risk Modelling

1 Motivation

In the analysis of multidimensional data, we are often interested in understanding the dependence structure of the marginals in order to build measures to be used, for example, in risk management. A classical instrument in the context of linear representations of dependence is the analysis of correlation structures. In this sense, a method often used in applications is factorial analysis, where data are assumed to satisfy a linear model. This approach is justified when data have a multivariate Gaussian distribution, whose dependence structure, as is well known, is uniquely

Matteo Grigoletto, Department of Statistical Sciences, University of Padua, e-mail: matteo.grigoletto@unipd.it

determined by correlation. Nevertheless, many data sets have characteristics contradicting the assumption of normality.

In many applications, the dependence in extremes is a very important issue (Coles, Heffernan and Tawn, 1999). In risk management, a correct understanding of joint extreme losses of a portfolio is essential. This problem requires a different approach with respect to the usual hypotheses on the asymptotic multivariate distribution of the extremes and the most relevant aspect of this paper will be the use of independent components analysis (ICA) (cf. Hyvärinen, Karhunen, and Oja, 2001) in order to measure extreme dependence. Differently from copula analysis, this approach does not require knowledge of the multivariate extreme-value distribution which could have generated the sample data. Chen, Härdle and Spokoiny (2005) use a similar method to analyse the volatility of financial series and measure the value-at-risk.

The paper is organised as follows. In Sections 2 and 3 we discuss models for multivariate extreme values based on copula functions, while in Section 4 the analysis of independent components with extreme-value sources is introduced. In Section 5 the Copula and ICA models are applied to the estimation of the value-at-risk for extreme returns relating to a portfolio of stock indices and to a portfolio of assets.

2 Models for extreme values

Let us consider the i.i.d. random variables $\tilde{X}_1, \dots, \tilde{X}_n$ and define $M_n = \max\{\tilde{X}_1, \dots, \tilde{X}_n\}$. The well known Fisher-Tippett theorem (Fisher and Tippett, 1928) states that, if there exists a sequence of constants $\{a_n > 0\}$ and $\{b_n\}$ such that $P\{(M_n - b_n)/a_n \leq x\} \rightarrow F(x)$, as $n \rightarrow \infty$, with F being a nondegenerate distribution function, then F is a member of the generalised extreme value (GEV) family of distributions, defined by

$$F(x; \theta) = \exp \left[- \left\{ 1 + \xi \frac{x - \mu}{\sigma} \right\}_+^{-1/\xi} \right], \quad (1)$$

where $\{s\}_+ = \max(s, 0)$ for any s , with parameters $\theta = (\mu, \sigma, \xi)$ satisfying $\sigma > 0$ and $\mu, \xi \in \mathbb{R}$. We denote distribution (1) by $\text{GEV}(\mu, \sigma, \xi)$.

Very often, focus is on the marginal return level z_α , defined by $F(z_\alpha; \theta) = 1 - \alpha$, for $0 \leq \alpha \leq 1$, so that $z_\alpha = \mu - \sigma[1 - \{-\log(1 - \alpha)\}^{-\xi}]/\xi$.

Now, let $(\tilde{X}_{1,i}, \dots, \tilde{X}_{p,i})$, for $i = 1, \dots, n$, and some known p , be p -variate i.i.d. random variables. The multivariate extreme value distributions arise as the limit distributions of normalised componentwise maxima. Define $M_{j,n} = \max\{\tilde{X}_{j,1}, \dots, \tilde{X}_{j,n}\}$, for $j = 1, \dots, p$ and let $\{a_{j,n} > 0\}$ and $\{b_{j,n}\}$ be suitable normalising sequences. Then, analogously to the univariate case,

$$P\{(M_{j,n} - b_{j,n})/a_{j,n} \leq x_j, j = 1, \dots, p\} \rightarrow F(x_1, \dots, x_p), \quad (2)$$

as $n \rightarrow \infty$, where F is nondegenerate in each margin. In particular, the marginal distributions must be generalised extreme value distributions. Under such linear normalisation, when it exists, and regulatory conditions (Reisnik, 1987), the joint dis-

tribution of the componentwise maxima also converges. Distributions in the class of possible nondegenerate limit distributions for linearly normalised componentwise maxima are known as multivariate extreme-value (MEV) distributions (Pickands, 1981). Here we are interested in modelling the dependence structure implied by the multivariate distribution function F in (2). The copula and independent component analysis methods will be used with this aim.

3 The copula approach

In statistics literature the idea of a copula arose as early as the 19th century in the context of discussions of non-normality in multivariate cases. Modern theories about copulas can be dated to about forty years ago when Sklar (1959) defined and provided some fundamental properties of a copula.

A copula is a joint distribution function of standard uniform random variables:

$$C(u_1, \dots, u_p) = P\{U_1 \leq u_1, \dots, U_p \leq u_p\},$$

where $U_i \sim U(0, 1)$, $i = 1, \dots, p$. Now, let F denote a p -dimensional distribution function with continuous margins $F_1, \dots, F_p$. Then, there exists a unique copula function C given by

$$C(u_1, \dots, u_p) = F(F_1^{-1}(u_1), \dots, F_p^{-1}(u_p))$$

with $F_i^{-1}(u) = \inf\{x : F_i(x) \geq u\}$. This result is known as Sklar's theorem. It can be easily proven that, when C is a copula function and $F_1, \dots, F_p$ are arbitrary distribution functions, then $F(x_1, \dots, x_p) = C(F_1(x_1), \dots, F_p(x_p))$ is a multivariate distribution function with marginals $F_1, \dots, F_p$. Therefore, copulas allow to characterise the dependence structure of a set of random variables separately from their marginal distributions.

An important feature of this result is that the marginal distributions do not need to be in any way similar to each other, nor is the choice of copula constrained by the choice of marginal distributions.

3.1 Copula families

In the present paper some commonly used families of copulas will be considered. In particular, we will adopt the Gaussian and Student- t families of copulas. Besides, we will use copulas in the Archimedean family, such as the Clayton, Frank and Gumbel copulas. The class of Archimedean copulas was named by Ling (1965), but it was recognised by Schweizer and Sklar (1961) in the study of t -norms. They are of interest mainly because, being non elliptical, allow modelling of a variety of different dependence structures. We consider, in particular, one-parameter Archimedean copulas.

Below, the families of copulas that will be considered later are described.

Gaussian copula For notational convenience, we set $u_i \equiv F_i(x_i)$. The Gaussian (or normal) copula is the copula of the multivariate normal distribution which is defined as

$$C_{\text{Gaussian}}(u_1, \dots, u_p; \rho) = \Phi_\rho(\Phi^{-1}(u_1), \dots, \Phi^{-1}(u_p)),$$

where Φ_ρ is a multi-dimensional standard Gaussian distribution function, with correlation matrix ρ , and Φ is the standard univariate Gaussian distribution function.

Student- t copula The t copula is based on the multivariate t distribution in the same way as the Gaussian copula is derived from the multivariate normal. The Student- t copula is defined by:

$$C_t(u_1, \dots, u_p; \rho, \nu) = t_{\rho, \nu}(t_\nu^{-1}(u_1), \dots, t_\nu^{-1}(u_p)),$$

The t copula is mostly useful because the degree of tail dependence can be set by changing the degrees of freedom parameter, ν . Large values for ν imply an approximate Gaussian distribution. Conversely, small values for ν increase tail dependence.

Clayton copula The Clayton family, first proposed by Clayton (1978), is defined as

$$C_{\text{Clayton}}(u_1, \dots, u_p; \theta) = \left(\sum_{i=1}^p u_i^{-\theta} - p + 1 \right)^{-1/\theta},$$

where $\theta > 0$.

Frank copula The Frank family, which appeared in Frank (1979), is discussed at length in Genest (1987). The Frank copula is defined by

$$C_{\text{Frank}}(u_1, \dots, u_p; \theta) = \frac{-1}{\theta} \log \left(1 + \frac{\prod_{i=1}^p (e^{-\theta u_i} - 1)}{(e^{-\theta} - 1)^{p-1}} \right),$$

where $\theta > 0$ when $p \geq 3$.

Gumbel copula The Gumbel family was introduced by Gumbel (1960). Since it has been discussed in Hougaard (1986), it is also known as the Gumbel-Hougaard family. This family is defined by

$$C_{\text{Gumbel}}(u_1, \dots, u_p; \theta) = \exp \left[- \left(\sum_{i=1}^p (-\log u_i)^\theta \right)^{1/\theta} \right],$$

where $\theta > 1$.

The fit of these copulas to financial data will be examined in the application section.

A MEV distribution can be characterised by its margins and a copula. It can be shown (Galambos, 1987) that copulas of a limiting MEV distribution satisfy

$$C(u_1^t, \dots, u_p^t) = C^t(u_1, \dots, u_p). \quad (3)$$

Conversely, every distribution function F defined through

$$F(x_1, \dots, x_p) = C(F_1(x_1), \dots, F_p(x_p)),$$

with C satisfying (3), is a MEV distribution, as soon as the functions F_i , $i = 1, \dots, p$, are univariate EV distributions. Therefore, copulas fulfilling property (3) are called MEV copulas. For example, the Gumbel copula described above is the only MEV copula in the family of Archimedean copulas.

4 Independent component analysis

In order to model componentwise maxima it is common to assume that a MEV copula is a good approximation for finite sample sizes. However, no parametric model for C can cover all the possible forms of MEV copulas. This lack of uniqueness sometimes prevents the application of extreme value theory to multivariate data, because it is necessary to choose the C copula (for an application to environmental data see Dupuis, 2005, and references therein).

In the next section we show an alternative approach, based on independent component analysis (ICA), which does not require the choice of a particular parametric form for the dependence structure.

In independent component analysis, the observed variable $x \in \mathbb{R}^p$ is modelled as $x = Vy$, where the underlying factors or sources, $y \in \mathbb{R}^p$, are independent random variables and V is an invertible $p \times p$ *mixing* matrix. ICA can also be seen as a linear transformation of observed variables x to a new set of independent variables $y = Wx$. The *demixing* matrix W is the inverse of the *mixing* matrix V .

ICA can be placed in a maximum likelihood framework. In fact, the likelihood function of the observations can be obtained from the densities of the sources using $f_x(x) = f_y(y)/|J|$, where f indicates the density function and J is the Jacobian of the transformation. Since in ICA the transformation is linear and sources are independent, we have $|J| = 1/|\det(W)|$ and $f_y(y) = \prod_{i=1}^p f_{y_i}(y_i)$, where f_{y_i} is the density function of the i th source component. These results lead us to write the log-likelihood as a function of $W = (w_1, \dots, w_p)'$, giving

$$\log f_x(x) = \log |\det(W)| + \sum_{i=1}^p \log f_{y_i}(w_i'x). \quad (4)$$

The covariance matrix for x can be immediately obtained as $\Sigma_x = VDV'$, where $D = \text{diag}(\sigma_1^2, \dots, \sigma_p^2)$, and σ_i^2 , $i = 1, \dots, p$, are the variances of the independent sources. The dependence structure of the ICA model is identified by means of the correlation matrix R_x obtained from Σ_x .

In the present case, we will assume the sources to have EV distributions. Explicit derivation of the resulting likelihood is then easily accomplished.

Table 1 Summary statistics for extreme returns

	DJI	SP500	NASDAQ	NYSE
Minimum	0.3519	0.2155	0.1196	0.2913
Maximum	25.6315	22.8997	12.0432	21.2863
1st Quartile	1.1427	1.0894	1.0432	1.0188
3rd Quartile	2.1580	2.1895	2.6110	2.0544
Mean	1.8847	1.8522	2.0630	1.7563
Median	1.6043	1.5341	1.6491	1.4442
Stdev	1.6025	1.5620	1.5840	1.4835
Skewness	8.1050	6.6994	2.4134	6.4609
Kurtosis	106.0580	74.8347	8.3988	69.5028
Spearman correlation				
DJI	1.0000	0.9350	0.5862	0.8519
SP500	0.9350	1.0000	0.6752	0.9088
NASDAQ	0.5862	0.6752	1.0000	0.6117
NYSE	0.8519	0.9088	0.6117	1.0000
Kendall correlation				
DJI	1.0000	0.7838	0.4170	0.6801
SP500	0.7838	1.0000	0.4931	0.7861
NASDAQ	0.4170	0.4931	1.0000	0.4430
NYSE	0.6801	0.7861	0.4430	1.0000

Portfolio DSNN: block = 1 month; n. obs. = 457

5 Data and empirical results

In this section Monte Carlo estimation of extreme risk for two portfolios is presented. Copula and ICA dependence models will be adopted.

5.1 Data

The first portfolio, denoted as DSNN, is formed by monthly maxima of daily negative log-returns concerning the indices Dow Jones Industrial Average (DJI), S&P 500 (SP500), NASDAQ composite (NASDAQ) and NYSE Composite (NYSE). Observations for these indices go from January, 3, 1972, to December, 29, 2009. The four indices ($p = 4$) are part of the CRSP (Center for Research in Security Prices) market groups of securities. Descriptive statistics and the Spearman ρ and Kendall τ correlation matrices among the extreme returns are listed in Table 1.

The second portfolio, denoted as AIGC, considers the monthly maxima of daily negative log-returns concerning the stock prices of the following assets: Alcoa Inc. Common Stock (AA), International Business Machines Corp. (IBM), General Electric Co. (GE) and Chevron Corp. (CVX). The data refer to the period from January, 2, 1972 to December, 29, 2009. These four assets ($p = 4$) are components of the Dow Jones index. Table 2 shows some descriptive statistics for this portfolio.

Table 2 Summary statistics for extreme returns

	AA	IBM	GE	CVX
Minimum	0.6529	0.7051	0.0000	0.5764
Maximum	27.5706	26.8193	19.1055	18.1948
1st Quartile	2.4118	1.7323	1.9268	1.9608
3rd Quartile	4.4402	3.4439	3.5091	3.5718
Mean	3.8207	2.9423	3.0666	3.0087
Median	3.1003	2.4065	2.5975	2.6766
Stdev	2.5805	2.1794	2.0035	1.7084
Skewness	3.6826	4.6836	2.8789	3.3991
Kurtosis	21.3282	37.5911	13.1455	20.7024
Spearman correlation				
AA	1.0000	0.3138	0.4105	0.3037
IBM	0.3138	1.0000	0.4232	0.2718
GE	0.4105	0.4232	1.0000	0.3503
CVX	0.3037	0.2718	0.3503	1.0000
Kendall correlation				
AA	1.0000	0.2142	0.2873	0.2077
IBM	0.2142	1.0000	0.2939	0.1871
GE	0.2873	0.2939	1.0000	0.2413
CVX	0.2077	0.1871	0.2413	1.0000

Portfolio AIGC: block = 1 month; n. obs. = 457

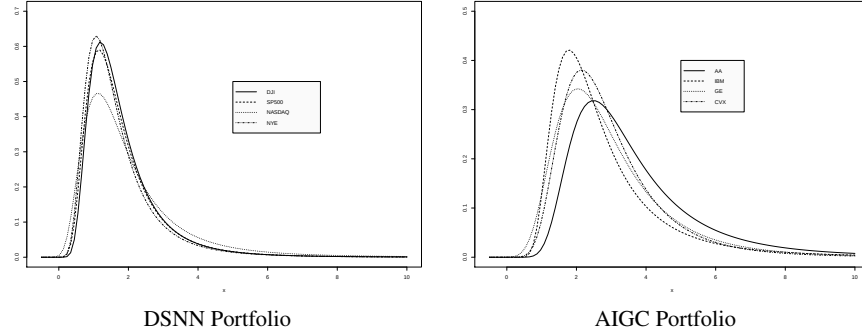
Table 3 lists the maximum likelihood estimates (robust standard deviation estimates in parentheses) of the GEV distributions for each portfolio component. In Figure 1 the corresponding density functions are shown.

Table 3 Maximum likelihood parameter estimates

Parameters	DJI	SP500	NASDAQ	NYSE	AA	IBM	GE	CVX
Location $\hat{\mu}$	1.3200 (0.0322)	1.2758 (0.0333)	1.3136 (0.0437)	1.1985 (0.0315)	2.7413 (0.0617)	2.0210 (0.0490)	2.2058 (0.0560)	2.2766 (0.0511)
Scale $\hat{\beta}$	0.6157 (0.0259)	0.6380 (0.0266)	0.8155 (0.0363)	0.6004 (0.0255)	1.1869 (0.0498)	0.9133 (0.0416)	1.0886 (0.0431)	0.9764 (0.0392)
Shape $\hat{\xi}$	0.2325 (0.0347)	0.2271 (0.0340)	0.2724 (0.0420)	0.2456 (0.0361)	0.2400 (0.0340)	0.3131 (0.0425)	0.1678 (0.0302)	0.1454 (0.0336)

Robust standard deviation estimates are reported in parentheses

It is interesting to note that the DSNM portfolio is characterised by a strong dependence structure and similar shape parameters for the GEV marginals. On the other hand, in the AIGC portfolio dependence is weaker and the shape parameters are more diversified.

Fig. 1 Estimated GEV distributions for extreme returns

5.2 Copula estimation

The copulas described in Section 3 have been estimated for the two portfolios, using the IFM (Inference for Margins) method (see Joe, 1997, Chapter 10). With this method parameters are estimated in two steps: 1) MLE estimates of the marginal GEV parameters (θ_1 , say) are obtained (see the last section); 2) MLE estimates of the copula parameters (θ_2 , say) are computed maximising the log-likelihood, and are therefore defined as

$$\hat{\theta}_2 = \text{ArgMax}_{\theta_2} \left\{ \sum_{i=1}^n \ln C(F_1(x_{1,i}), \dots, F_p(x_{p,i}); \theta_2, \hat{\theta}_1) \right\},$$

where $F_j(x_j)$, $j = 1, \dots, p$ are the GEV marginals parametrised with $\hat{\theta}_1$. The IFM MLE estimates will be denoted by $\hat{\theta}_{\text{IFM}} = (\hat{\theta}_1, \hat{\theta}_2)'$.

Table 5 shows MLE estimates for the five copulas considered, while Table 4 shows the values of the Akaike and Cramér-von Mises statistics S_n defined in equation (2) of Genest, Remillard and Beaudoin (2009). Small values of these statistics suggest a higher reliability of extreme returns for a copula with respect to the others.

Table 4 Akaike and Cramér-von Mises statistics

Copula	Portfolio DSNN		Portfolio AIGC	
	AIC	S_n	AIC	S_n
Normal	-4.2914	0.1162	-0.6815	0.0933
Student- t	-4.8780	0.1109	-0.7171	0.0853
Clayton	-2.0885	2.7761	-0.3549	0.3893
Frank	-3.0460	0.3282	-0.5927	0.0912
Gumbel	-3.2284	0.3801	-0.7316	0.0459

Probably as a consequence of the different dependence structure, we note smaller values of AIC and S_n for the DSNN portfolio when the Student- t elliptical copula is

chosen; on the other hand, these criteria suggest the choice of the Gumbel copula for the AIGC portfolio.

Table 5 MLE estimates of copula parameters for the DSNN and AIGC portfolios

	DSNN					AIGC				
	Elliptical copula		Archimidean copula			Elliptical copula		Archimidean copula		
	Normal	Student- t	Clayton	Frank	Gumbel	Normal	Student- t	Clayton	Frank	Gumbel
$\hat{\theta}$			1.2198	6.9332	2.1784			0.3454	2.1882	1.2917
$\hat{\rho}_{21}$	0.9394	0.9374				0.3156	0.3257			
$\hat{\rho}_{31}$	0.5991	0.5955				0.4571	0.4625			
$\hat{\rho}_{32}$	0.8405	0.8945				0.3509	0.3439			
$\hat{\rho}_{41}$	0.6747	0.6757				0.4521	0.4357			
$\hat{\rho}_{42}$	0.8871	0.9549				0.2903	0.2849			
$\hat{\rho}_{43}$	0.6115	0.6499				0.3920	0.3888			
$\hat{\nu}$		3.7619					12.8231			

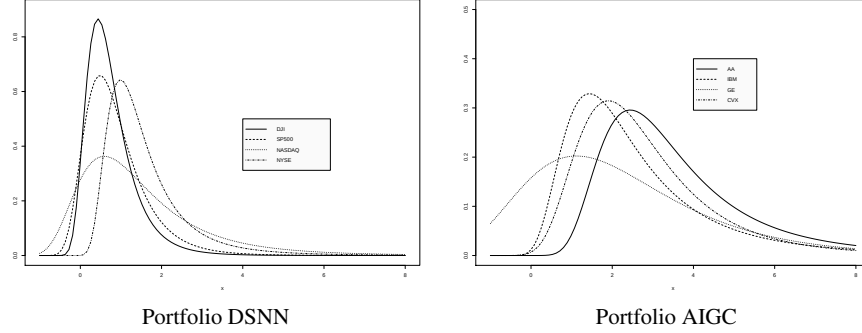
5.3 Estimation of the ICA Model

The ICA model described in Section 4 will be estimated, for the DSNN and AIGC portfolios, by maximising the log-likelihood function (4), using GEV sources.

Table 6 displays the estimated parameters (robust standard deviation estimates are in parentheses) for the GEV sources for each component and the demixing matrices for the two portfolios. The estimated densities are shown in Figure 2.

Table 6 ICA estimates for the DSNN and AIGC portfolios

Parameters		DJI	SP500	NASDAQ	NYSE	AA	IBM	GE	CHX
Location	$\hat{\mu}$	0.4694 (0.1151)	0.5198 (0.1562)	0.7310 (0.1021)	1.1010 (0.0867)	2.6928 (0.0424)	1.7123 (0.0519)	1.1791 (0.0709)	2.0396 (0.0487)
Scale	$\hat{\beta}$	0.4265 (0.1241)	0.5591 (0.1274)	1.0247 (0.0498)	0.5853 (0.0618)	1.2732 (0.0565)	1.1588 (0.0613)	1.8197 (0.0261)	1.1782 (0.0357)
Shape	$\hat{\xi}$	0.0884 (0.0647)	0.0653 (0.0921)	0.1527 (0.1122)	0.2212 (0.0695)	0.2238 (0.0881)	0.2789 (0.0425)	0.0341 (0.0538)	0.1235 (0.0686)
Demixing matrix Portfolio DSNN					Demixing matrix Portfolio AIGC				
		0.8113	0.7053	0.0294	0.3697	0.9370	0.0126	0.0380	0.0422
		0.3662	0.9498	0.0978	0.3625	0.0958	0.7510	0.0576	0.1105
		0.2789	0.3765	0.7156	0.2726	0.2434	0.1183	0.5937	0.1551
		-0.1751	0.2836	0.0828	0.9324	0.1408	0.0200	0.0555	0.7930

Fig. 2 ICA density estimates for the GEV sources**Fig. 3** Estimated GEV sources for the ICA model

5.4 Risk measures

In order to evaluate the extremal risk implied by the DSNN and AIGC portfolios, 10000 Monte Carlo samples with length equal to those of the observed series have been extracted. Then, the ICA-GEV model and the Student- t (DSNN portfolio) and Gumbel (GEV portfolio) copulas, with GEV marginals, have been estimated.

From the simulated data, two risk measures have been computed: the value at risk (VaR) and the expected shortfall (ES) (see McNeil, Frey and Embrechts, 2005, for a definition). The estimated values of VaR and ES are reported in Tables 7 and 8.

Two probability levels are considered, respectively corresponding to 12 months and 36 months waiting periods.

From these results, it is clear that the ICA-GEV and copula approaches lead to similar evaluations of the risk for the portfolios.

However, limits in the applicability of the considered methods arise from the non-uniqueness of the chosen copula and the latent representation of the multivariate distribution implied by the ICA-GEV model. This is especially problematic when larger portfolios are considered.

For this reason, we think that future research should be directed towards a simultaneous use of the two approaches, based on the evaluation of the distance between the multivariate distribution introduced through a particular selected copula and the latent distribution in the ICA model (see, e.g., Abayomi, Lall and de la Pena, 2007).

Acknowledgements

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Table 7 VaR and ES with 10000 Monte-Carlo simulations for the DSNN portfolio

		<i>t</i> Copula				ICA Model			
		VaR		ES		VaR		ES	
		12m	36m	12m	36m	12m	36m	12m	36m
DJI	1st Quartile	3.2146	4.4353	4.5159	5.9039	3.0190	3.6877	3.6717	4.3121
	3rd Quartile	3.4447	4.9404	5.0068	6.9601	3.1588	3.9334	3.8867	4.7018
	Mean	3.3340	4.7027	4.7784	6.4906	3.0907	3.8156	3.7823	4.5208
	Median	3.3261	4.6723	4.7423	6.3825	3.0876	3.8078	3.7753	4.4984
	Stdev	0.1754	0.3830	0.3840	0.8395	0.1051	0.1834	0.1610	0.2996
SP500	1st Quartile	3.2224	4.4686	4.5429	5.9611	3.0282	3.7503	3.7368	4.4393
	3rd Quartile	3.4641	4.9831	5.0484	7.0158	3.1810	4.0232	3.9713	4.8668
	Mean	3.3472	4.7434	4.8121	6.5398	3.1074	3.8931	3.8587	4.6666
	Median	3.3377	4.7134	4.7840	6.4465	3.1034	3.8817	3.8498	4.6393
	Stdev	0.1803	0.3895	0.3884	0.8394	0.1124	0.2007	0.1745	0.3258
NASDAQ	1st Quartile	3.9428	5.7515	5.9090	8.0580	3.6703	4.8333	4.8568	6.0771
	3rd Quartile	4.2903	6.5222	6.7117	9.7851	3.9033	5.2870	5.2683	6.8820
	Mean	4.1233	6.1681	6.3452	9.0413	3.7898	5.0719	5.0718	6.5181
	Median	4.1112	6.1219	6.2875	8.8445	3.7847	5.0481	5.0537	6.4625
	Stdev	0.2566	0.5898	0.6272	1.4297	0.1726	0.3443	0.3146	0.6303
NYSE	1st Quartile	3.0746	4.3053	4.3975	5.8250	2.9829	4.0217	4.0873	5.2684
	3rd Quartile	3.3081	4.8325	4.9128	6.9113	3.1834	4.4623	4.5118	6.1642
	Mean	3.1967	4.5839	4.6752	6.4403	3.0862	4.2546	4.3163	5.7689
	Median	3.1881	4.5545	4.6419	6.3336	3.0812	4.2260	4.2929	5.6875
	Stdev	0.1762	0.3950	0.4032	0.8941	0.1505	0.3264	0.3274	0.7150

Table 8 VaR and ES with 10000 Monte-Carlo simulations for the AIGC portfolio

		Gumbel Copula				ICA Model			
		VaR		ES		VaR		ES	
		12m	36m	12m	36m	12m	36m	12m	36m
DJI	1st Quartile	6.4279	8.8211	8.9826	11.7457	6.3988	8.7009	8.8368	11.4564
	3rd Quartile	6.8933	9.8350	10.0097	13.9115	6.8536	9.6517	9.7700	13.3874
	Mean	6.6707	9.3609	9.5350	12.9512	6.6324	9.2069	9.3397	12.5373
	Median	6.6521	9.3033	9.4687	12.7499	6.6162	9.1560	9.2830	12.3354
	Stdev	0.3477	0.7581	0.7727	1.6999	0.3343	0.7206	0.7255	1.5833
SP500	1st Quartile	5.1333	7.4090	7.6715	10.4919	4.9876	6.9502	7.1293	9.4848
	3rd Quartile	5.5555	8.4189	8.7666	12.9298	5.3506	7.8040	8.0076	11.4083
	Mean	5.3555	7.9498	8.2755	11.9097	5.1764	7.4049	7.6158	10.5945
	Median	5.3413	7.8944	8.1820	11.5628	5.1593	7.3572	7.5415	10.3603
	Stdev	0.3158	0.7618	0.8713	2.0713	0.2726	0.6425	0.6971	1.6160
NASDAQ	1st Quartile	5.2941	7.0974	7.1429	9.0581	5.2351	6.5368	6.4902	7.7214
	3rd Quartile	5.6544	7.7983	7.8108	10.4250	5.5148	7.0171	6.9007	8.4357
	Mean	5.4776	7.4667	7.4943	9.8002	5.3789	6.7843	6.7034	8.1024
	Median	5.4666	7.4325	7.4639	9.7009	5.3728	6.7661	6.6910	8.0630
	Stdev	0.2646	0.5423	0.5065	1.0398	0.2045	0.3563	0.3062	0.5484
NYSE	1st Quartile	4.9744	6.4878	6.5169	8.1030	4.9134	6.2788	6.2803	7.6883
	3rd Quartile	5.2770	7.0856	7.0613	9.1692	5.1879	6.8058	6.7562	8.5929
	Mean	5.1303	6.8017	6.7975	8.6772	5.0544	6.5546	6.5294	8.1688
	Median	5.1199	6.7754	6.7778	8.6055	5.0480	6.5297	6.5111	8.1093
	Stdev	0.2271	0.4520	0.4095	0.8191	0.2047	0.3950	0.3532	0.6860

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