





# Dependence of Stock Returns in Bull and Bear Markets

Jadran Dobrić, Gabriel Frahm, and Friedrich Schmid

**Abstract** Despite of its many shortcomings, Pearson's rho is typically used in finance literature for measuring the dependence structure of stock returns. A conditional version of Spearman's rho is suggested as an alternative dependence measure. This approach is purely nonparametric and avoids any kind of model misspecification. Hypothesis tests for the conditional rank-correlation coefficient in bull and bear markets are derived and their finite-sample performance is studied by Monte Carlo simulation. Further, the daily returns of stocks contained in the German stock index DAX 30 are analyzed. The empirical study reveals significant differences in the dependence of stock returns in bull and bear markets.

**Key words:** Bear market, bootstrapping, bull market, conditional Spearman's rho, copulas, Monte Carlo simulation, Pearson's rho, stock returns.

## 1 Introduction

The standard estimator for the linear correlation coefficient according to Karl Pearson still seems to be the most commonly used measure of dependence of two random

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variables  $X$  and  $Y$  though its many shortcomings have been often documented (see, e.g., Embrechts et al., 2002). Pearson's rho is strongly affected by the marginal distributions of  $X$  and  $Y$  and its estimates are sensitive to outliers (Embrechts et al., 2002). Further, it only quantifies linear dependence though *monotone dependence* is often much more relevant. The random variables  $X$  and  $Y$  are said to possess a strong monotone dependence if there exist two real-valued and strictly increasing functions  $f$  and  $g$  such that  $|\text{Corr}(f(X), g(Y))|$  is large.

It is easy to construct dependence structures where the linear correlation coefficient of  $X$  and  $Y$  is close to 0 but even so one can find two monotone transformations  $f$  and  $g$  such that  $\text{Corr}(f(X), g(Y)) = 1$ . Consider for instance the random variables  $X = e^Z$  and  $Y = e^{\sigma Z}$  with  $\sigma > 0$  and  $Z \sim \mathcal{N}(0, 1)$  (McNeil et al., 2005, p. 205). Since  $\text{Corr}(\log X, \log Y) = 1$  they possess a *perfect* monotone dependence structure, i.e.  $X$  and  $Y$  are *comonotonic* (Nelsen, 2006, p. 32). Nevertheless,  $\text{Corr}(X, Y)$  is a function of  $\sigma$  and can take any value between 0 ( $\sigma \rightarrow \infty$ ) and 1 ( $\sigma = 1$ ).

Copula theory and the dependence measures derived thereof are a convincing alternative to the linear correlation coefficient. Due to Sklar's theorem (Sklar, 1959) it is known that a joint distribution function can be split up into its *copula* (i.e. its dependence structure) and its marginal distributions. A meaningful dependence measure should be invariant under monotone transformations of the corresponding random variables. Examples of such measures are Spearman's rho, Kendall's tau, Gini's gamma, and Blomquist's beta. In this paper we confine ourselves to the rank-correlation coefficient or its corresponding estimator, i.e. Spearman's rho. For surveys on copulas and dependence measures see, e.g., Cherubini et al. (2004), Joe (1997), and Nelsen (2006).

We investigate the contemporaneous dependence of two stock returns  $X$  and  $Y$ . In particular, we concentrate on the question whether dependence is significantly different in bull and bear markets, i.e. in case of a joint upswing or downswing. This question and related problems have been already investigated in finance literature (see, e.g., Ang and Chen, 2002, Erb et al., 1994, Fortin and Kuzmics, 2002, Junker and May, 2005, Patton, 2004, Silvapulle and Granger, 2001, Vaz de Melo Mendes, 2005). But we think that the statistical methods, in particular the use of Pearson's rho is unsatisfactory. Hence, there is space for further contributions.

Bear and bull markets are characterized as follows. A bear market is present if the two stock returns  $X$  and  $Y$  contemporaneously fall short of the  $100p\%$  quantiles of their corresponding cumulative distribution functions. Analogously, a bull market is given whenever  $-X$  and  $-Y$  fall short of the corresponding  $100q\%$  quantiles. Here  $p$  and  $q$  have to be pre-determined. The lower  $p$ -quantile of the cumulative distribution function (c.d.f.) of a stock return is commonly known as the *value-at-risk* where  $p$  is the so-called *shortfall probability*. The value-at-risk is frequently used in risk management. So it seems to be a natural choice for characterizing bull and bear markets.

Our approach is purely nonparametric. Contrary to Patton (2004) and Vaz de Melo Mendes (2005) we do not fit specific copulas to the data. Specifying the copula by some parametric model can lead to erroneous conclusions if the chosen model is wrong. From our point of view it is not necessary to rely on the parametric approach

if the sample size is large enough. We are interested in financial data analysis and in that context it is easy to access many thousands of observations. By following the nonparametric approach we avoid any kind of model misspecification.

In this work we develop conditional versions of the rank-correlation coefficient to assess the dependence structure of stock returns in bull and bear markets. In contrast, some authors analyze the dependence structure of outliers in financial data by using the so-called *tail-dependence coefficient* (Fortin and Kuzmics, 2002, Junker and May, 2005). After applying parametric methods these authors come to the conclusion that ‘the empirical joint distribution of return pairs on stock indices displays high tail-dependence in the lower tail and low tail-dependence in the upper tail’ (Fortin and Kuzmics, 2002). Dobrić and Schmid (2005) as well as Frahm et al. (2005) found that estimating the tail-dependence coefficient by *nonparametric methods* can lead to very large estimation errors even if there are many observations. Hence the tail-dependence coefficient is not an appropriate alternative.

Though we focus on computational statistics and the empirical analysis of stock returns we have to introduce some statistical theory in order to have a formal basis for our testing procedure. This is done in Section 2, where some copula theory is presented. It allows a precise formulation of the null hypotheses to be tested. The testing procedure is described in Section 3. A Monte Carlo (MC) simulation is presented in Section 4 which shows that the procedure works well for sample sizes which are typically available in practice. In particular it is demonstrated that the test keeps the prescribed error probability of the first kind and has sufficient power to detect violations of the null hypothesis. In Section 5 we investigate the daily returns of stocks from the German stock index *DAX 30* between 1973-01-02 and 2008-11-14 and Section 6 concludes.

## 2 Some Copula Theory

In this section we introduce some notions from copula theory (Joe, 1997, Nelsen, 2006) which are required for understanding the testing procedure to be described below. Let  $X$  and  $Y$  be two random variables with joint c.d.f.  $F(x, y) = \mathbf{P}(X \leq x, Y \leq y)$  and marginal cumulative distribution functions  $G(x) = \mathbf{P}(X \leq x)$  and  $H(y) = \mathbf{P}(Y \leq y)$  for all  $x, y \in \mathbb{R}$ . The quantile functions with respect to  $G$  and  $H$  are given by  $G^{-1}(p) = \inf\{x: G(x) \geq p\}$  and  $H^{-1}(p) = \inf\{y: H(y) \geq p\}$  for  $0 \leq p \leq 1$ .

Throughout this paper we assume that  $G$  and  $H$  are absolutely continuous functions. Therefore, according to Sklar’s theorem (Sklar, 1959), there exists a unique copula  $C: [0, 1]^2 \rightarrow [0, 1]$  such that

$$F(x, y) = C(G(x), H(y)), \quad \forall x, y \in \mathbb{R}.$$

The function  $C$  is the joint c.d.f. of  $U = G(X)$  and  $V = H(Y)$ . The rank-correlation coefficient of  $X$  and  $Y$  is given by

$$\rho := \mathbb{C}\text{orr}(U, V) = 12 \int_{[0,1]^2} uv dC(u, v) - 3 = 12 \int_{[0,1]^2} C(u, v) d(u, v) - 3. \quad (1)$$

See Nelsen (2006, p. 167) for the latter representation of  $\rho$ .

For every fixed  $p$  with  $0 < p < 1$  we define

$$A_L := \left\{ (x, y) : x \leq G^{-1}(p), y \leq H^{-1}(p) \right\}.$$

In the following we assume that  $\mathbf{P}\{(X, Y) \in A_L\} = C(p, p) > 0$ . Consider the *conditional* joint c.d.f.

$$\begin{aligned} F_L(x, y) &:= \mathbf{P}(X \leq x, Y \leq y | (X, Y) \in A_L) = \frac{F(x \wedge G^{-1}(p), y \wedge H^{-1}(p))}{F(G^{-1}(p), H^{-1}(p))} \\ &= \frac{C(G(x \wedge G^{-1}(p)), H(y \wedge H^{-1}(p)))}{C(p, p)}, \quad \forall x, y \in \mathbb{R}. \end{aligned}$$

The corresponding conditional marginal distribution functions are given by

$$\begin{aligned} G_L(x) &:= \mathbf{P}(X \leq x | (X, Y) \in A_L) = F_L(x, H^{-1}(p)) \\ &= \frac{C(G(x \wedge G^{-1}(p)), p)}{C(p, p)}, \quad \forall x \in \mathbb{R}, \end{aligned}$$

and  $H_L(y)$  respectively. Since  $G_L$  and  $H_L$  are absolutely continuous, according to Sklar's theorem there exists also a unique copula  $C_L : [0, 1]^2 \rightarrow [0, 1]$  such that

$$F_L(x, y) = C_L(G_L(x), H_L(y)), \quad \forall x, y \in \mathbb{R}.$$

Indeed, Juri and Wthrich (2002) call

$$C_L(u, v) = F_L(G_L^{-1}(u), H_L^{-1}(v)), \quad \forall u, v \in [0, 1],$$

the *extreme tail dependence copula relative to C at the level p*. Instead we will call  $C_L$  *lower tail-copula* and the phrase ‘relative to  $C$  at the level  $p$ ’ will be usually dropped for convenience.

Using the lower tail-copula we now can define the lower conditional rank-correlation coefficient, viz

$$\rho_L = 12 \int_{[0,1]^2} uv dC_L(u, v) - 3.$$

In the empirical part of this work this measures the monotone dependence of stock returns  $X$  and  $Y$  conditional on the bear market  $(X, Y) \in A_L$ .

An analogue definition can be found for the *upper tail-copula*  $C_U$ . This is the lower tail-copula relative to the *survival copula* according to  $C$  (Nelsen, 2006, Section 2.6), i.e.

$$\bar{C}(u, v) := u + v - 1 + C(1 - u, 1 - v), \quad \forall u, v \in [0, 1],$$

at the level  $q$  ( $0 < q < 1$ ). The survival copula simply corresponds to the copula of  $(-X, -Y)$  and thus  $C_U$  is the copula of  $(-X, -Y)$  under the condition that  $(-X, -Y) \in A_U$ . Here the area  $A_U$  is calculated similarly to  $A_L$  just by using the quantile functions of  $-X$  and  $-Y$  at  $q$  rather than the quantile functions of  $X$  and  $Y$  at  $p$ . Hence, the upper conditional rank-correlation coefficient  $\rho_U$  measures the monotone dependence of stock returns in a bull market. In the following we will have to guarantee that  $A_L \cap A_U = \emptyset$  which is equivalent to  $p + q \leq 1$ .

In most cases it is not possible to derive the conditional copulas  $C_L$  or  $C_U$  in closed form. Therefore  $\rho_L$  and  $\rho_U$  cannot be calculated explicitly. However, MC simulation is a convenient tool for obtaining numerical approximations to  $\rho_L$  and  $\rho_U$  with sufficient precision. We apply this method to calculate the conditional rank-correlation coefficients for the Gauss-,  $t_3$ -, Clayton-, and Gumbel-copula (see Table 1 and Table 2). The Gauss- and  $t_3$ -copula are given by

$$C_{\text{Gauss}}(u, v; \theta) = \Phi_{\theta}(\Phi^{-1}(u), \Phi^{-1}(v)), \quad \forall u, v \in [0, 1],$$

where

$$\Phi_{\theta}(x, y) := \int_{-\infty}^x \int_{-\infty}^y \frac{1}{2\pi\sqrt{1-\theta^2}} \cdot \exp\left(-\frac{s^2 - 2\theta st + t^2}{2(1-\theta^2)}\right) ds dt$$

as well as

$$C_{t_3}(u, v; \theta) = t_{3,\theta}(t_3^{-1}(u), t_3^{-1}(v)), \quad \forall u, v \in [0, 1],$$

with

$$t_{3,\theta}(x, y) = \int_{-\infty}^x \int_{-\infty}^y \frac{1}{2\pi\sqrt{1-\theta^2}} \cdot \left(1 + \frac{s^2 - 2\theta st + t^2}{3(1-\theta^2)}\right)^{-\frac{5}{2}} ds dt,$$

where  $t_3$  denotes Student's univariate  $t$ -distribution function with 3 degrees of freedom and  $-1 < \theta < 1$ . Note that the linear correlation coefficient is symbolized by the parameter  $\theta$  rather than  $\rho$ . This is because to avoid possible confusions with the (unconditional) rank-correlation coefficient of  $C_{\text{Gauss}}$  or  $C_{t_3}$ . The unconditional rank-correlation coefficient of the Gauss-copula corresponds to  $\rho = 6/\pi \cdot \arcsin(\theta/2)$  (Hult and Lindskog, 2002). To our knowledge there exists no such closed-form expression for the  $t_3$ -copula.

The Clayton-copula is given by

$$C_{\text{Clayton}}(u, v; \theta) = (u^{-\theta} + v^{-\theta} - 1)^{-1/\theta}, \quad \forall u, v \in [0, 1],$$

with  $\theta \geq 0$ . In the limiting case  $\theta = 0$  the Clayton-copula corresponds to the *independence* or *product copula*  $\Pi(u, v) := uv$  (Nelsen, 2006, p. 11).

The Gumbel-copula can be written as

$$C_{\text{Gumbel}}(u, v; \theta) = \exp\left[-\{(-\log u)^{\theta} + (-\log v)^{\theta}\}^{1/\theta}\right], \quad \forall u, v \in [0, 1],$$

| Gauss-copula |                  |                  |                  |                  |                  |                  |
|--------------|------------------|------------------|------------------|------------------|------------------|------------------|
|              | $\theta = 0.25$  |                  | $\theta = 0.50$  |                  | $\theta = 0.75$  |                  |
| $p = q$      | lower            | upper            | lower            | upper            | lower            | upper            |
| 0.05         | .0404<br>(.0004) | .0407<br>(.0004) | .1109<br>(.0003) | .1114<br>(.0003) | .2622<br>(.0002) | .2624<br>(.0002) |
| 0.20         | .0601<br>(.0001) | .0601<br>(.0001) | .1595<br>(.0001) | .1593<br>(.0001) | .3485<br>(.0001) | .3483<br>(.0001) |
| 0.35         | .0775<br>(.0001) | .0774<br>(.0001) | .1972<br>(.0001) | .1973<br>(.0001) | .4090<br>(.0001) | .4091<br>(.0001) |
| 0.50         | .0962<br>(.0001) | .0962<br>(.0001) | .2354<br>(.0001) | .2356<br>(.0001) | .4655<br>(.0000) | .4656<br>(.0000) |

| $t_3$ -copula |                  |                  |                  |                  |                  |                  |
|---------------|------------------|------------------|------------------|------------------|------------------|------------------|
|               | $\theta = 0.25$  |                  | $\theta = 0.50$  |                  | $\theta = 0.75$  |                  |
| $p = q$       | lower            | upper            | lower            | upper            | lower            | upper            |
| 0.05          | .3373<br>(.0003) | .3369<br>(.0003) | .4043<br>(.0002) | .4044<br>(.0002) | .5264<br>(.0002) | .5265<br>(.0002) |
| 0.20          | .3186<br>(.0001) | .3183<br>(.0001) | .3968<br>(.0001) | .3967<br>(.0001) | .5361<br>(.0001) | .5361<br>(.0001) |
| 0.35          | .2984<br>(.0001) | .2984<br>(.0001) | .3913<br>(.0001) | .3913<br>(.0001) | .5484<br>(.0001) | .5485<br>(.0001) |
| 0.50          | .2756<br>(.0001) | .2756<br>(.0001) | .3882<br>(.0001) | .3882<br>(.0001) | .5651<br>(.0000) | .5652<br>(.0000) |

**Table 1** MC approximations to  $\rho_L$  and  $\rho_U$  for the Gauss- and  $t_3$ -copula possessing different values for  $\theta$ . We use  $N_{MC} = 1000$  MC replications, each one generating a sample from the corresponding copula with sample size  $n = 10^6$ . The standard errors of the approximations are given in parentheses.

with  $\theta \geq 1$ . Note that for  $\theta = 1$  once again the independence copula evolves. The values for  $\theta$  in Table 2 are chosen such that the *unconditional* rank-correlation coefficient corresponds to  $\rho = 0.3, 0.5, 0.7$ . The relationship between  $\theta$  and  $\rho$  can be obtained by numerical integration or MC simulation (see Joe, 1997, p. 147).

For our approximations of the conditional rank-correlation coefficients given in Table 1 and Table 2 we used  $N_{MC} = 1000$  MC replications. In each one a sample from  $C$  with sample size  $n = 10^6$  has been generated. Both for the simulation study and for the empirical study following later on we set  $p = q$ . Only the Clayton-copula allows for a closed-form representation of  $C_L$ . If  $C$  is a Clayton-copula the lower tail-copula  $C_L$  is equal to  $C$  for any  $0 < p < 1$  (Juri and Wthrich, 2002). That means  $\rho_L$  corresponds to the unconditional rank-correlation coefficient  $\rho$ .

The null hypothesis we are going to test can be formalized as

$$H_0 : \rho_L = \rho_U \\ \text{vs. } H_1 : \rho_L \neq \rho_U,$$

where some  $p$  and  $q$  with  $p + q \leq 1$  are fixed. In the present framework  $H_1$  implies that the monotone dependence of stock returns in bear markets is not the same as in bull markets.

Instead of a two-sided hypothesis test, a one-sided test like

$$H_0 : \rho_L \leq \rho_U \\ \text{vs. } H_1 : \rho_L > \rho_U$$

| Clayton-copula |                  |                  |                  |                  |                  |                  |
|----------------|------------------|------------------|------------------|------------------|------------------|------------------|
|                | $\theta = 0.51$  |                  | $\theta = 1.08$  |                  | $\theta = 2.13$  |                  |
| $p = q$        | lower            | upper            | lower            | upper            | lower            | upper            |
| 0.05           | .3004<br>(.0002) | .0025<br>(.0005) | .5001<br>(.0002) | .0018<br>(.0004) | .7002<br>(.0001) | .0035<br>(.0004) |
| 0.20           | .3003<br>(.0001) | .0040<br>(.0001) | .4999<br>(.0001) | .0113<br>(.0001) | .7000<br>(.0001) | .0318<br>(.0001) |
| 0.35           | .3001<br>(.0001) | .0130<br>(.0001) | .4999<br>(.0001) | .0356<br>(.0001) | .7000<br>(.0000) | .0906<br>(.0001) |
| 0.50           | .3001<br>(.0001) | .0298<br>(.0001) | .5000<br>(.0000) | .0764<br>(.0001) | .7000<br>(.0000) | .1783<br>(.0001) |

| Gumbel-copula |                  |                  |                  |                  |                  |                  |
|---------------|------------------|------------------|------------------|------------------|------------------|------------------|
|               | $\theta = 1.26$  |                  | $\theta = 1.54$  |                  | $\theta = 2.07$  |                  |
| $p = q$       | lower            | upper            | lower            | upper            | lower            | upper            |
| 0.05          | .0319<br>(.0004) | .3499<br>(.0002) | .0697<br>(.0003) | .4504<br>(.0002) | .1431<br>(.0003) | .5849<br>(.0001) |
| 0.20          | .0515<br>(.0001) | .3158<br>(.0001) | .1106<br>(.0001) | .4392<br>(.0001) | .2206<br>(.0001) | .5871<br>(.0001) |
| 0.35          | .0697<br>(.0001) | .2906<br>(.0001) | .1476<br>(.0001) | .4314<br>(.0001) | .2843<br>(.0001) | .5916<br>(.0000) |
| 0.50          | .0912<br>(.0001) | .2744<br>(.0001) | .1885<br>(.0001) | .4276<br>(.0001) | .3507<br>(.0000) | .5990<br>(.0000) |

**Table 2** MC approximations to  $\rho_L$  and  $\rho_U$  for the Clayton- and Gumbel-copula possessing different values for  $\theta$ . We use  $N_{MC} = 1000$  MC replications, each one generating a sample from the corresponding copula with sample size  $n = 10^6$ . The standard errors of the approximations are given in parentheses.

is of general interest, since  $H_1$  implies that the monotone dependence is higher in bear markets than in bull markets.

The null hypothesis  $H_0: \rho_L = \rho_U$  stated above might be also of interest in another context. Both in theory and application of copulas it is sometimes questionable whether the random vector  $(X, Y)$  is *radially symmetric* or not (Nelsen, 2006, Section 2.7). Radial symmetry is a useful property which guarantees that  $\rho_L = \rho_U$  for all  $0 < p < 1$  since  $C$  and the corresponding survival copula coincide. In order to test the null hypothesis  $H'_0$ : ‘The random vector  $(X, Y)$  is radially symmetric’, one can apply the two-sided test and reject  $H'_0$  if  $H_0$  is rejected.

### 3 The Testing Procedure

In this section we describe the testing procedure. The first part requires independent and identically distributed (i.i.d.) data. It is well-known that short-term asset returns typically exhibit strong patterns of serial dependence. However, the i.i.d. assumption may serve as an appropriate starting point and there might exist several applications beyond financial data analysis where this assumption is adequate. Therefore it is worth to illustrate the testing procedure in the i.i.d. case. Afterwards we will drop the assumption of serially independent asset returns and explain how the test can be modified to account for the purpose of time series analysis.

### 3.1 Independent and Identically Distributed Data

Let  $(X, Y), (X_1, Y_1), \dots, (X_n, Y_n)$  be i.i.d. As we do not assume that the marginal cumulative distribution functions  $G$  and  $H$  are known, we have to estimate them by

$$\hat{G}_n(x) = \frac{1}{n} \cdot \sum_{i=1}^n \mathbf{1}_{\{X_i \leq x\}} \quad \text{and} \quad \hat{H}_n(y) = \frac{1}{n} \cdot \sum_{i=1}^n \mathbf{1}_{\{Y_i \leq y\}}.$$

The corresponding estimate of the quantile function  $G^{-1}$  is given by

$$\hat{G}_n^{-1}(p) = \inf\{x: \hat{G}_n(x) \geq p\}$$

and  $\hat{H}_n^{-1}(p)$  respectively. For some fixed  $p$  and  $q$  with  $p + q \leq 1$  we can define

$$\hat{A}_L := \{(x, y): x \leq \hat{G}_n^{-1}(p), y \leq \hat{H}_n^{-1}(p)\}$$

and  $\hat{A}_U$  in the same manner. We also define the sample sizes  $n_L := |\hat{A}_L|$  with respect to the lower left and  $n_U := |\hat{A}_U|$  with respect to the upper right area of the empirical copula (here  $|\cdot|$  denotes the cardinality of a set). The observations in  $\hat{A}_L$  and  $\hat{A}_U$  can be used for estimating  $\rho_L$  and  $\rho_U$ . More precisely,

$$\hat{\rho}_{L,n} = \frac{12}{n_L} \cdot \sum_{i \in \mathbf{I}_{\hat{A}_L}} \frac{r_{L,n}(X_i)}{n_L} \cdot \frac{r_{L,n}(Y_i)}{n_L} - 3,$$

where  $\mathbf{I}_{\hat{A}_L}$  denotes the set of indices  $i$  such that  $(X_i, Y_i) \in \hat{A}_L$ . Although  $\hat{\rho}_{L,n}$  is calculated on the basis of  $n_L$  data points, for notational convenience the number of observations is indicated by  $n$  rather than  $n_L$  (this is adequate since  $n_L$  depends on  $n$ ).

Further,  $r_{L,n}(\cdot)$  is the rank of a marginal observation relative to all observations in  $\hat{A}_L$ . Note that  $r_{L,n}(X_i)/n_L = \hat{G}_{L,n}(X_i)$  and  $r_{L,n}(Y_i)/n_L = \hat{H}_{L,n}(Y_i)$ . Here  $\hat{G}_{L,n}$  is the empirical counterpart of  $G_L$ , i.e.

$$\hat{G}_{L,n}(x) = \frac{\hat{C}_n(\hat{G}_n(x \wedge \hat{G}_n^{-1}(p)), p)}{\hat{C}_n(p, p)}, \quad \forall x \in \mathbb{R},$$

where

$$\hat{C}_n(u, v) := \frac{1}{n} \cdot \sum_{i=1}^n \mathbf{1}_{\{r_n(X_i)/n \leq u\}} \mathbf{1}_{\{r_n(Y_i)/n \leq v\}}, \quad \forall u, v \in [0, 1],$$

represents the empirical copula. Moreover,  $\hat{H}_{L,n}$  is defined respectively. The definition of the estimator  $\hat{\rho}_{U,n}$  follows immediately, just by using the survival copula according to  $\hat{C}_n$  (which is determined by the observations in the upper right area  $\hat{A}_U$ ).

Schmid and Schmidt (2006) have already shown that Spearman's rho is consistent and asymptotically normally distributed. The same holds for the conditional versions of Spearman's rho described above, i.e.

$$\sqrt{n_L}(\hat{\rho}_{L,n} - \rho_L) \xrightarrow{d} \mathcal{N}(0, \sigma_L^2) \quad \text{and} \quad \sqrt{n_U}(\hat{\rho}_{U,n} - \rho_U) \xrightarrow{d} \mathcal{N}(0, \sigma_U^2)$$

as  $n_L, n_U \rightarrow \infty$ , provided the lower left and upper right tail-copulas exist.

**Proposition 1.** *Let the distribution of  $(X, Y)$  be absolutely continuous. Suppose that the partial derivatives of the corresponding copula  $C$  exist and are continuous, too. Further, define  $\Delta\hat{\rho}_n := \hat{\rho}_{L,n} - \hat{\rho}_{U,n}$  and  $\Delta\rho := \rho_L - \rho_U$  with shortfall probabilities  $p, q > 0$  such that  $p + q \leq 1$ . If  $C(p, p), \bar{C}(q, q) > 0$  then*

$$\sqrt{n}(\Delta\hat{\rho}_n - \Delta\rho) \xrightarrow{d} \mathcal{N}\left(0, \frac{\sigma_L^2}{C(p, p)} + \frac{\sigma_U^2}{\bar{C}(q, q)}\right), \quad n \rightarrow \infty.$$

*Proof.* Note that  $n_L/n \xrightarrow{\text{a.s.}} C(p, p)$  as  $n \rightarrow \infty$ . That means

$$\sqrt{n}(\hat{\rho}_{L,n} - \rho_L) = \sqrt{\frac{n}{n_L}} \sqrt{n_L}(\hat{\rho}_{L,n} - \rho_L) \xrightarrow{d} \mathcal{N}\left(0, \frac{\sigma_L^2}{C(p, p)}\right), \quad n \rightarrow \infty,$$

and the corresponding result holds also for  $\hat{\rho}_{U,n}$ . Since  $p + q \leq 1$ , the considered tails of the copula do not overlap. That means  $\hat{\rho}_{L,n}$  and  $\hat{\rho}_{U,n}$  are stochastically independent. This leads immediately to the asymptotic variance given in the proposition.

In practical situations  $p$  and  $q$  have to be sufficiently large such that  $n_L$  and  $n_U$  do not become too small. A typical rule of thumb might be given by  $n_L, n_U \geq 40$ . Suppose for the moment that  $C$  corresponds to the product copula. In that case it is expected to meet  $p^2n$  observations in the lower left part of the empirical copula. That means the shortfall probabilities should be such that  $p, q \geq \sqrt{40/n}$ . E.g. for the sample size  $n = 1000$  (that means an observation period of approximately 4 years)  $p$  and  $q$  should be not smaller than 0.2. Admittedly, financial data cannot be appropriately described by the product copula since in most cases there is some sort of positive dependence between stock returns. So there are even *more* observations in the corresponding corners of the empirical copula. Thus our rule of thumb guarantees that there are always enough data for large sample inferences.

The asymptotic variances  $\sigma_L^2$  and  $\sigma_U^2$  depend on the tail-copulas  $C_L$  and  $C_U$ . In general they cannot be calculated explicitly (Schmid and Schmidt, 2006). The same holds for the asymptotic variance of  $\Delta\hat{\rho}_n$ , i.e.  $\tau^2 := \sigma_L^2/C(p, p) + \sigma_U^2/\bar{C}(q, q)$ . However, the latter can be approximated by a simple bootstrap. For conducting the hypothesis test one has to choose an appropriate significance level  $\alpha > 0$  as well as the shortfall probabilities  $p > 0$  and  $q > 0$  such that  $p + q \leq 1$ . Now the test procedure reads as follows:

1. Compute  $\hat{\rho}_{L,n}$  and  $\hat{\rho}_{U,n}$  from the observations in  $\hat{A}_L$  and  $\hat{A}_U$ .
2. Compute  $N_B$  bootstrap replications from the entire sample. For each replication calculate  $\hat{\rho}_{L,n}$  and  $\hat{\rho}_{U,n}$  as well as the corresponding difference  $\Delta\hat{\rho}_n$ .

3. Estimate the asymptotic variance  $\tau^2$  of  $\Delta\hat{\rho}_n$  from the bootstrap and calculate the test statistic  $T = \Delta\hat{\rho}_n / \sqrt{\hat{\tau}^2/n}$ , where  $\hat{\tau}^2$  is the bootstrap estimate of  $\tau^2$ .
- 4a. Reject  $H_0: \rho_L = \rho_U$  if

$$|T| \geq \Phi^{-1}\left(1 - \frac{\alpha}{2}\right),$$

where  $\Phi$  denotes the standard normal c.d.f.

The one-sided hypothesis tests differ only in the fourth step from the two-sided test:

- 4b. Reject  $H_0: \rho_L \leq \rho_U$  or  $H_0: \rho_L \geq \rho_U$  if  $T \geq \Phi^{-1}(1 - \alpha)$  or  $T \leq \Phi^{-1}(\alpha)$ .

### 3.2 Serially Dependent Data

Now let  $(X_1, Y_1), \dots, (X_n, Y_n)$  be a sample from a stationary process  $\{(X_t, Y_t)\}_{t \in \mathbb{Z}}$ . It is supposed that the process exhibits a *weak dependence structure*, i.e. the two one-sided processes  $\{(X_t, Y_t)\}_{t \leq 0}$  and  $\{(X_t, Y_t)\}_{t \geq l}$  are asymptotically independent as  $l \rightarrow \infty$  (Politis, 2003). This condition is sometimes referred to as the strong mixing or  $\alpha$ -mixing condition and it can be shown that many time series models frequently used in theory and practice meet that requirement.

One of the referees pointed out that it is important to account for serial dependence, since  $\hat{\rho}_{L,n}$  and  $\hat{\rho}_{U,n}$  are no longer independent in that case. For instance, in periods of great turbulence the lower and upper conditional Spearman's rho might be strongly correlated. However, it can be assumed that  $\Delta\hat{\rho}_n$  remains asymptotically normally distributed under the weak dependence assumption of time series analysis. That means

$$\sqrt{n}(\Delta\hat{\rho}_n - \Delta\rho) \xrightarrow{d} \mathcal{N}(0, \tau_{LR}^2), \quad n \rightarrow \infty, \quad (2)$$

where  $\tau_{LR}^2$  represents the *long-run variance* of  $\Delta\hat{\rho}_n$ . This assumption seems natural, since the weak convergence property of Spearman's rho is based on the weak convergence of an empirical copula process (Schmid and Schmidt, 2006). By using a weak form of the central limit theorem from time series analysis one can argue that the weak convergence property is still satisfied under the strong mixing condition.

There exist many possible techniques for estimating the long-run variance  $\tau_{LR}^2$  of the statistic  $\Delta\hat{\rho}_n$ , such as subsampling or block-bootstrapping (Politis, 2003). It has been shown by Politis et al. (2001) that the subsampling procedure leads to consistent estimates of the long-run variance under very mild regularity conditions. However, subsampling is probably not the best choice in our setting. The reason is that for getting an unbiased estimate for  $\tau_{LR}^2$ , the number of observations within each subsample must be considerably small relative to the overall sample size. Note that in our context only a small part of each subsample can be used for calculating  $\hat{\rho}_{L,n}$  and  $\hat{\rho}_{U,n}$  but for a proper approximation of the long-run variance it has to be guaranteed also that each subsample contains a sufficiently large number of usable observations.

Thus we will concentrate on a block-bootstrap procedure suggested by Knsch (1989). Consider a block length  $b$  with  $0 < b < n$  and the corresponding  $n - b + 1$  blocks, i.e.

$$B_i = \{(X_{i+1}, Y_{i+1}), \dots, (X_{i+b}, Y_{i+b})\}, \quad i = 0, 1, \dots, n - b.$$

One bootstrap replication from the entire sample is given by drawing  $k = \lfloor n/b \rfloor$  blocks with replacement and concatenating these blocks to form a new pseudo-series of asset returns. Note that the series consists of  $l = kb \approx n$  pseudo-observations. For each of the  $N_B$  bootstrap replications one can calculate  $\Delta \hat{\rho}_l$  and the long-run variance can be estimated from the given  $N_B$  realizations. Finally, the test statistic is given by

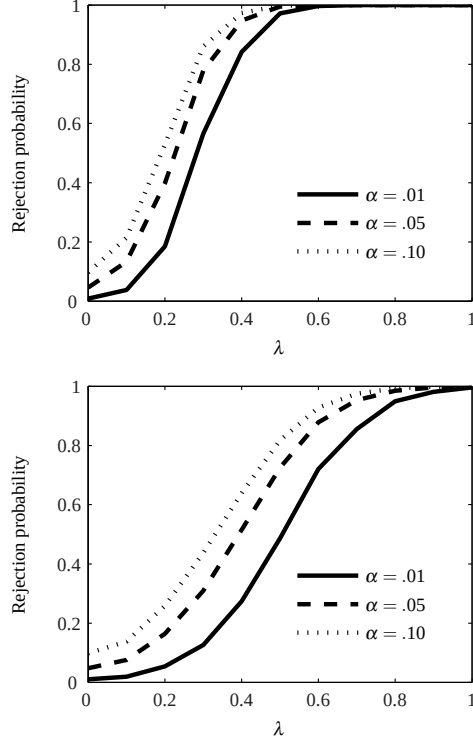
$$T = \frac{\Delta \hat{\rho}_n}{\sqrt{\hat{\tau}_{LR}^2/n}},$$

where  $\hat{\tau}_{LR}^2$  is the estimated long-run variance. Under the weak convergence property (2) and if  $b \rightarrow \infty$  and  $n/b \rightarrow \infty$ , the estimator  $\hat{\tau}_{LR}^2$  is consistent for  $\tau_{LR}^2$ . Hence,  $T$  can be used in the same way as the test statistic discussed in Section 3.1.

## 4 Finite-Sample Properties

In this section we investigate the finite-sample properties of the testing procedure described in Section 3.1. The results are obtained by MC simulations for various special cases. These are essentially defined by the copula under study. First we are interested in the rejection probability of the procedure if  $H_0: \rho_L = \rho_U$  is true and  $\alpha$  is the prescribed error probability of the first kind. We consider the Gauss- and  $t_3$ -copula which belong to the class of elliptical copulas. Elliptical copulas are radially symmetric which means that the aforementioned null hypotheses is true. The selected values for the copula parameter are  $\theta = 0.25, 0.5, 0.75$ , the values for  $p$  are given by  $p = 0.2, 0.35, 0.5$ , and we validate the error probabilities  $\alpha = 0.01, 0.05, 0.1$ . The simulated sample size is  $n = 2500$  (i.e. approximately 10 trading years), the number of bootstrap replications amounts to  $N_B = 1000$ , and the number of MC replications is  $N_{MC} = 1000$ . The results of the simulations are summarized in Panel 1 of Table 3. We can see that the approximated rejection probabilities satisfactorily agree with the prescribed error probabilities.

We are also interested in the *power* of the testing procedure, i.e. the probability of rejection if  $H_0$  is wrong. For that purpose we consider the Clayton- and the Gumbel-copula. It is well-known that these copulas are not radially symmetric and thus in general  $\rho_L \neq \rho_U$ . Remember that the parameter  $\theta$  of both copula families (cf. p. 7) has been selected in such a way that the unconditional rank-correlation coefficients are equal to  $\rho = 0.3, 0.5, 0.7$ . The results of the MC simulations are given in Panel 2 of Table 3. It can be seen that for every fixed  $p$  and  $\alpha$  the power is an increasing



**Fig. 1** Power functions of the two-sided hypothesis test for the mixed copulas  $C_{\text{Mix1}}$  (left hand side) and  $C_{\text{Mix2}}$  (right hand side) as a function of  $\lambda$ . The results are obtained by MC simulation for the sample size  $n = 2500$ ,  $N_B = 1000$  bootstrap replications, and  $N_{MC} = 10000$  MC replications using the shortfall probability  $p = q = 0.5$ .

function of  $\theta$ . This is because the asymmetry of the Archimedean copulas  $C_{\text{Clayton}}$  and  $C_{\text{Gumbel}}$  increases with  $\theta$  (cf. Nelsen, 2006, Ch. 4).

Similar results are obtained for the two one-sided tests which can be taken from Table 4 and Table 5. The rejection probabilities become very large whenever  $H_1$  is true. In contrast, if  $H_0$  is true our simulations produce no false rejection. For instance, consider the right-sided test  $H_0: \rho_L \leq \rho_U$  vs.  $H_1: \rho_L > \rho_U$ . In that case the null hypothesis is fulfilled for the Gumbel-copula. Panel 2 of Table 4 shows that there is no rejection for any given unconditional rank-correlation coefficient  $\rho$ , shortfall probability  $p$ , and error probability  $\alpha$ . In contrast, for the Clayton-copula the alternative hypothesis is true and consequently the rejection probabilities are very high (e.g. roughly 90% for  $\rho = 0.3$ ,  $p = 0.2$ , and  $\alpha = 0.1$ ). Moreover, for  $\rho = 0.5$  and  $\rho = 0.7$ ,  $H_0$  is rejected for the Clayton-copula in almost every simulated case.

Now we want to investigate the relationship between asymmetry and power. For that purpose we consider the mixed copula

| $H_0: \rho_L = \rho_U$ vs. $H_1: \rho_L \neq \rho_U$ |          |                 |                 |                  |                  |                  |                  |
|------------------------------------------------------|----------|-----------------|-----------------|------------------|------------------|------------------|------------------|
| Panel 1                                              |          | $\theta = 0.25$ |                 | $\theta = 0.50$  |                  | $\theta = 0.75$  |                  |
| $p = q$                                              | $\alpha$ | Gauss           | $t_3$           | Gauss            | $t_3$            | Gauss            | $t_3$            |
| 0.20                                                 | 0.10     | .091<br>(.0091) | .083<br>(.0087) | .081<br>(.0086)  | .085<br>(.0088)  | .091<br>(.0091)  | .093<br>(.0092)  |
|                                                      | 0.05     | .043<br>(.0064) | .047<br>(.0067) | .039<br>(.0061)  | .041<br>(.0063)  | .048<br>(.0068)  | .048<br>(.0068)  |
|                                                      | 0.01     | .008<br>(.0028) | .011<br>(.0033) | .006<br>(.0024)  | .007<br>(.0026)  | .011<br>(.0033)  | .011<br>(.0033)  |
| 0.35                                                 | 0.10     | .106<br>(.0097) | .081<br>(.0086) | .092<br>(.0091)  | .108<br>(.0098)  | .095<br>(.0093)  | .087<br>(.0089)  |
|                                                      | 0.05     | .057<br>(.0073) | .038<br>(.0060) | .049<br>(.0068)  | .053<br>(.0071)  | .048<br>(.0068)  | .051<br>(.0070)  |
|                                                      | 0.01     | .015<br>(.0038) | .009<br>(.0030) | .011<br>(.0033)  | .007<br>(.0026)  | .006<br>(.0024)  | .013<br>(.0036)  |
| 0.50                                                 | 0.10     | .104<br>(.0097) | .088<br>(.0090) | .088<br>(.0090)  | .113<br>(.0100)  | .089<br>(.0090)  | .098<br>(.0094)  |
|                                                      | 0.05     | .060<br>(.0075) | .043<br>(.0064) | .035<br>(.0058)  | .056<br>(.0073)  | .048<br>(.0068)  | .049<br>(.0068)  |
|                                                      | 0.01     | .019<br>(.0043) | .011<br>(.0033) | .008<br>(.0028)  | .008<br>(.0028)  | .006<br>(.0024)  | .011<br>(.0033)  |
| Panel 2                                              |          | $\rho = 0.30$   |                 | $\rho = 0.50$    |                  | $\rho = 0.70$    |                  |
| $p = q$                                              | $\alpha$ | Clayton         | Gumbel          | Clayton          | Gumbel           | Clayton          | Gumbel           |
| 0.20                                                 | 0.10     | .815<br>(.0123) | .752<br>(.0137) | 1.000<br>(.0000) | .965<br>(.0058)  | 1.000<br>(.0000) | .999<br>(.0010)  |
|                                                      | 0.05     | .715<br>(.0143) | .635<br>(.0152) | .999<br>(.0010)  | .938<br>(.0076)  | 1.000<br>(.0000) | .996<br>(.0020)  |
|                                                      | 0.01     | .456<br>(.0158) | .371<br>(.0153) | .993<br>(.0026)  | .800<br>(.0126)  | 1.000<br>(.0000) | .999<br>(.0010)  |
| 0.35                                                 | 0.10     | .988<br>(.0034) | .926<br>(.0083) | 1.000<br>(.0000) | .999<br>(.0010)  | 1.000<br>(.0000) | 1.000<br>(.0000) |
|                                                      | 0.05     | .981<br>(.0043) | .876<br>(.0104) | 1.000<br>(.0000) | .995<br>(.0022)  | 1.000<br>(.0000) | 1.000<br>(.0000) |
|                                                      | 0.01     | .928<br>(.0082) | .704<br>(.0144) | 1.000<br>(.0000) | .980<br>(.0044)  | 1.000<br>(.0000) | 1.000<br>(.0000) |
| 0.50                                                 | 0.10     | .999<br>(.0010) | .974<br>(.0050) | 1.000<br>(.0000) | 1.000<br>(.0000) | 1.000<br>(.0000) | 1.000<br>(.0000) |
|                                                      | 0.05     | .999<br>(.0010) | .945<br>(.0072) | 1.000<br>(.0000) | 1.000<br>(.0000) | 1.000<br>(.0000) | 1.000<br>(.0000) |
|                                                      | 0.01     | .996<br>(.0020) | .837<br>(.0117) | 1.000<br>(.0000) | .995<br>(.0022)  | 1.000<br>(.0000) | 1.000<br>(.0000) |

**Table 3** MC approximations of the rejection probabilities for the Gauss- and  $t_3$ -copula (Panel 1) and for the Clayton- and Gumbel-copula (Panel 2) given  $H_0: \rho_L = \rho_U$ . The simulated sample size is  $n = 2500$ , the number of bootstrap replications corresponds to  $N_B = 1000$ , and the number of MC replications is  $N_{MC} = 1000$ . The standard errors for the approximated rejection probabilities are given in parentheses.

$$C_{\text{Mix1}}(u, v; \lambda, \theta_0, \theta_1) := \lambda C_{\text{Clayton}}(u, v; \theta_1) + (1 - \lambda) C_{\text{Gauss}}(u, v; \theta_0),$$

where  $0 \leq \lambda \leq 1$ . Further, the copula parameters  $\theta_0, \theta_1$  are such that the unconditional rank-correlation coefficients of  $C_{\text{Clayton}}(u, v; \theta_1)$  and  $C_{\text{Gauss}}(u, v; \theta_0)$  correspond to  $\rho = 0.5$ . Hence, the mixed copula possesses the same unconditional rank-correlation coefficient for every  $\lambda$  (see the formula for  $\rho$  on p. 6).

Note that  $\rho_L = \rho_U$  is true for the Gauss-copula but for the Clayton-copula it holds that  $\rho_L > \rho_U$  and so the mixing parameter  $\lambda$  determines the degree of asymmetry given by  $C_{\text{Mix1}}(u, v; \lambda, \theta_0, \theta_1)$ . If one considers the two-sided hypothesis test with

| $H_0: \rho_L \leq \rho_U$ vs. $H_1: \rho_L > \rho_U$ |          |                  |                 |                  |                 |                  |                 |
|------------------------------------------------------|----------|------------------|-----------------|------------------|-----------------|------------------|-----------------|
| Panel 1                                              |          | $\theta = 0.25$  |                 | $\theta = 0.50$  |                 | $\theta = 0.75$  |                 |
| $p = q$                                              | $\alpha$ | Gauss            | $t_3$           | Gauss            | $t_3$           | Gauss            | $t_3$           |
| 0.20                                                 | 0.10     | .091<br>(.0091)  | .096<br>(.0093) | .099<br>(.0094)  | .095<br>(.0093) | .096<br>(.0093)  | .099<br>(.0094) |
|                                                      | 0.05     | .047<br>(.0067)  | .041<br>(.0063) | .040<br>(.0062)  | .041<br>(.0063) | .047<br>(.0067)  | .049<br>(.0068) |
|                                                      | 0.01     | .009<br>(.0030)  | .007<br>(.0026) | .006<br>(.0024)  | .007<br>(.0026) | .013<br>(.0036)  | .009<br>(.0030) |
| 0.35                                                 | 0.10     | .103<br>(.0096)  | .093<br>(.0092) | .086<br>(.0089)  | .099<br>(.0094) | .097<br>(.0094)  | .094<br>(.0092) |
|                                                      | 0.05     | .053<br>(.0071)  | .049<br>(.0068) | .038<br>(.0060)  | .044<br>(.0065) | .047<br>(.0067)  | .044<br>(.0065) |
|                                                      | 0.01     | .012<br>(.0034)  | .010<br>(.0031) | .008<br>(.0028)  | .006<br>(.0024) | .008<br>(.0028)  | .011<br>(.0033) |
| 0.50                                                 | 0.10     | .109<br>(.0099)  | .092<br>(.0091) | .103<br>(.0096)  | .110<br>(.0099) | .086<br>(.0860)  | .094<br>(.0092) |
|                                                      | 0.05     | .050<br>(.0069)  | .046<br>(.0066) | .046<br>(.0066)  | .053<br>(.0071) | .049<br>(.0068)  | .054<br>(.0071) |
|                                                      | 0.01     | .015<br>(.0038)  | .011<br>(.0033) | .011<br>(.0033)  | .011<br>(.0033) | .010<br>(.0031)  | .008<br>(.0028) |
| Panel 2                                              |          | $\rho = 0.30$    |                 | $\rho = 0.50$    |                 | $\rho = 0.70$    |                 |
| $p = q$                                              | $\alpha$ | Clayton          | Gumbel          | Clayton          | Gumbel          | Clayton          | Gumbel          |
| 0.20                                                 | 0.10     | .899<br>(.0095)  | .000<br>(.0000) | 1.000<br>(.0000) | .000<br>(.0000) | 1.000<br>(.0000) | .000<br>(.0000) |
|                                                      | 0.05     | .815<br>(.0123)  | .000<br>(.0000) | 1.000<br>(.0000) | .000<br>(.0000) | 1.000<br>(.0000) | .000<br>(.0000) |
|                                                      | 0.01     | .574<br>(.0156)  | .000<br>(.0000) | .997<br>(.0017)  | .000<br>(.0000) | 1.000<br>(.0000) | .000<br>(.0000) |
| 0.35                                                 | 0.10     | .997<br>(.0017)  | .000<br>(.0000) | 1.000<br>(.0000) | .000<br>(.0000) | 1.000<br>(.0000) | .000<br>(.0000) |
|                                                      | 0.05     | .988<br>(.0034)  | .000<br>(.0000) | 1.000<br>(.0000) | .000<br>(.0000) | 1.000<br>(.0000) | .000<br>(.0000) |
|                                                      | 0.01     | .961<br>(.0061)  | .000<br>(.0000) | 1.000<br>(.0000) | .000<br>(.0000) | 1.000<br>(.0000) | .000<br>(.0000) |
| 0.50                                                 | 0.10     | 1.000<br>(.0000) | .000<br>(.0000) | 1.000<br>(.0000) | .000<br>(.0000) | 1.000<br>(.0000) | .000<br>(.0000) |
|                                                      | 0.05     | .999<br>(.0010)  | .000<br>(.0000) | 1.000<br>(.0000) | .000<br>(.0000) | 1.000<br>(.0000) | .000<br>(.0000) |
|                                                      | 0.01     | .999<br>(.0010)  | .000<br>(.0000) | 1.000<br>(.0000) | .000<br>(.0000) | 1.000<br>(.0000) | .000<br>(.0000) |

**Table 4** MC approximations of the rejection probabilities for the Gauss- and  $t_3$ -copula (Panel 1) and for the Clayton- and Gumbel-copula (Panel 2) given  $H_0: \rho_L \leq \rho_U$ . The simulated sample size is  $n = 2500$ , the number of bootstrap replications corresponds to  $N_B = 1000$ , and the number of MC replications is  $N_{MC} = 1000$ . The standard errors for the approximated rejection probabilities are given in parentheses.

$H_0: \rho_L \leq \rho_U$ ,  $\lambda = 0$  means that the null hypothesis is true whereas the alternative hypothesis holds for every  $\lambda > 0$ . The larger  $\lambda$  the more often  $H_0$  should be rejected.

A similar result is obtained for the mixed copula

$$C_{\text{Mix2}}(u, v; \lambda, \theta_0, \theta_2) := \lambda C_{\text{Gumbel}}(u, v; \theta_2) + (1 - \lambda) C_{\text{Gauss}}(u, v; \theta_0),$$

where  $\theta_2$  is such that the rank-correlation coefficient associated with  $C_{\text{Gumbel}}(u, v; \theta_2)$  once again amounts to  $\rho = 0.5$ . The corresponding power functions are given in Figure 1. Both figures demonstrate that the hypothesis test always keeps the prescribed

| $H_0: \rho_L \geq \rho_U$ vs. $H_1: \rho_L < \rho_U$ |          |                 |                 |                 |                  |                 |                  |
|------------------------------------------------------|----------|-----------------|-----------------|-----------------|------------------|-----------------|------------------|
| Panel 1                                              |          | $\theta = 0.25$ |                 | $\theta = 0.50$ |                  | $\theta = 0.75$ |                  |
| $p = q$                                              | $\alpha$ | Gauss           | $t_3$           | Gauss           | $t_3$            | Gauss           | $t_3$            |
| 0.20                                                 | 0.10     | .093<br>(.0092) | .097<br>(.0094) | .089<br>(.0090) | .090<br>(.0090)  | .096<br>(.0093) | .096<br>(.0093)  |
|                                                      | 0.05     | .044<br>(.0065) | .042<br>(.0063) | .041<br>(.0063) | .044<br>(.0065)  | .044<br>(.0065) | .044<br>(.0065)  |
|                                                      | 0.01     | .009<br>(.0030) | .011<br>(.0033) | .007<br>(.0026) | .012<br>(.0034)  | .006<br>(.0024) | .013<br>(.0036)  |
| 0.35                                                 | 0.10     | .108<br>(.0098) | .083<br>(.0087) | .100<br>(.0095) | .111<br>(.0099)  | .096<br>(.0093) | .087<br>(.0089)  |
|                                                      | 0.05     | .053<br>(.0071) | .032<br>(.0056) | .054<br>(.0071) | .064<br>(.0077)  | .048<br>(.0068) | .043<br>(.0064)  |
|                                                      | 0.01     | .013<br>(.0036) | .007<br>(.0026) | .013<br>(.0036) | .012<br>(.0034)  | .005<br>(.0022) | .013<br>(.0036)  |
| 0.50                                                 | 0.10     | .095<br>(.0093) | .097<br>(.0094) | .103<br>(.0096) | .094<br>(.0092)  | .087<br>(.0089) | .088<br>(.0090)  |
|                                                      | 0.05     | .054<br>(.0071) | .042<br>(.0063) | .042<br>(.0063) | .060<br>(.0075)  | .040<br>(.0062) | .044<br>(.0065)  |
|                                                      | 0.01     | .016<br>(.0040) | .007<br>(.0026) | .004<br>(.0020) | .009<br>(.0030)  | .003<br>(.0017) | .015<br>(.0038)  |
| Panel 2                                              |          | $\rho = 0.30$   |                 | $\rho = 0.50$   |                  | $\rho = 0.70$   |                  |
| $p = q$                                              | $\alpha$ | Clayton         | Gumbel          | Clayton         | Gumbel           | Clayton         | Gumbel           |
| 0.20                                                 | 0.10     | .000<br>(.0000) | .856<br>(.0111) | .000<br>(.0000) | .986<br>(.0037)  | .000<br>(.0000) | 1.000<br>(.0000) |
|                                                      | 0.05     | .000<br>(.0000) | .752<br>(.0137) | .000<br>(.0000) | .965<br>(.0058)  | .000<br>(.0000) | .999<br>(.0010)  |
|                                                      | 0.01     | .000<br>(.0000) | .481<br>(.0158) | .000<br>(.0000) | .870<br>(.0106)  | .000<br>(.0000) | .994<br>(.0024)  |
| 0.35                                                 | 0.10     | .000<br>(.0000) | .964<br>(.0059) | .000<br>(.0000) | .999<br>(.0010)  | .000<br>(.0000) | 1.000<br>(.0000) |
|                                                      | 0.05     | .000<br>(.0000) | .926<br>(.0083) | .000<br>(.0000) | .999<br>(.0010)  | .000<br>(.0000) | 1.000<br>(.0000) |
|                                                      | 0.01     | .000<br>(.0000) | .790<br>(.0129) | .000<br>(.0000) | .987<br>(.0036)  | .000<br>(.0000) | 1.000<br>(.0000) |
| 0.50                                                 | 0.10     | .000<br>(.0000) | .991<br>(.0030) | .000<br>(.0000) | 1.000<br>(.0000) | .000<br>(.0000) | 1.000<br>(.0000) |
|                                                      | 0.05     | .000<br>(.0000) | .974<br>(.0050) | .000<br>(.0000) | 1.000<br>(.0000) | .000<br>(.0000) | 1.000<br>(.0000) |
|                                                      | 0.01     | .000<br>(.0000) | .892<br>(.0098) | .000<br>(.0000) | .996<br>(.0020)  | .000<br>(.0000) | 1.000<br>(.0000) |

**Table 5** MC approximations of the rejection probabilities for the Gauss- and  $t_3$ -copula (Panel 1) and for the Clayton- and Gumbel-copula (Panel 2) given  $H_0: \rho_L \geq \rho_U$ . The simulated sample size is  $n = 2500$ , the number of bootstrap replications corresponds to  $N_B = 1000$ , and the number of MC replications is  $N_{MC} = 1000$ . The standard errors for the approximated rejection probabilities are given in parentheses.

error probability of the first kind and the rejection probability indeed is an increasing function of the mixing parameter  $\lambda$ . Similar results can be obtained for other constellations of  $\rho$  and  $p$ .

|                                    | $p = 0.1$ | $p = 0.2$ | $p = 0.3$ | $p = 0.4$ | $p = 0.5$ |
|------------------------------------|-----------|-----------|-----------|-----------|-----------|
| $\overline{\hat{\rho}_{L,n}}$      | 39%       | 34%       | 34%       | 34%       | 35%       |
| $\overline{\hat{\rho}_{U,n}}$      | 34%       | 27%       | 25%       | 24%       | 24%       |
| $\overline{\Delta \hat{\rho}_n}$   | 5%        | 7%        | 9%        | 10%       | 11%       |
| $ \overline{\Delta \hat{\rho}_n} $ | 12%       | 10%       | 10%       | 11%       | 11%       |

**Table 6** Average conditional Spearman’s rhos, differences, and absolute differences of all 435 asset combinations for different shortfall probabilities  $p = q$ .

## 5 Empirical Results for German Stock Returns

Now we consider daily observations from 1973-01-02 to 2008-11-14 of the 30 stocks which are listed in the German stock index DAX 30. The stock prices have been adjusted for dividends, splits, etc. Our analyze is based on the daily log-returns of the assets (zero returns have been deleted). The maximum number of observations is given by  $n = 9359$  trading days. Table 6 contains the sample means of the upper and lower conditional Spearman’s rhos for all 435 asset combinations given the shortfall probabilities  $p = q = 0.1, 0.2, 0.3, 0.4, 0.5$ . Here  $\overline{\hat{\rho}_{L,n}}$  denotes the mean lower and  $\overline{\hat{\rho}_{U,n}}$  the mean upper conditional Spearman’s rho, whereas  $\overline{\Delta \hat{\rho}_n}$  is the mean difference and  $|\overline{\Delta \hat{\rho}_n}|$  the mean absolute difference between  $\hat{\rho}_{L,n}$  and  $\hat{\rho}_{U,n}$ . It can be seen that the lower conditional Spearman’s rhos are up to 11 points larger in average than the upper conditional Spearman’s rhos. However, without a meaningful theoretical argument it is not possible to judge whether this gap between bull and bear markets is rather ‘large’ or ‘small’ and we would like to avoid such kind of statements. Instead we will discuss how much of the empirical evidence leads to significant results in our hypothesis tests.

It is worth to point out that the outcomes of the test can depend substantially on the shortfall probability  $p$ . The upper part of Figure 2 shows the lower and upper conditional Spearman’s rho as a function of  $p$  for BASF vs. Henkel. The difference between the rhos (see the lower part of Figure 2) seems to be negligible if  $p \leq 0.25$  but it can be very large for  $p > 0.25$ . The lower right part of Figure 2 indicates that it is easy to find a suitable  $p$  such that  $H_0: \rho_L \leq \rho_U$  can be rejected on a significance level of  $\alpha = 0.05$ , although in fact there are not many statistical arguments in favor of  $H_1: \rho_L > \rho_U$ . As a counterexample consider Figure 3 giving the conditional Spearman’s rhos of BASF vs. Thyssen. There is only a small range for  $p$  where  $H_0$  cannot be rejected. That means there is a large amount of evidence for supporting  $H_1$  but this could be easily concealed by exploiting the data. Finally, Figure 4 contains the conditional Spearman’s rhos for BASF vs. Infineon. Only in that case data mining is impossible since there is no  $p$  for which the null hypothesis could be rejected. We conclude that the presented hypothesis tests work only if  $p$  is chosen before examining  $\hat{\rho}_{L,n}$  and  $\hat{\rho}_{U,n}$  with different shortfall probabilities. Otherwise the tests would seriously suffer from a selection bias.

### 5.1 Two-Sided Hypothesis Test

It is clear that the estimates  $\hat{\rho}_L$  and  $\hat{\rho}_U$  are different from each other for every combination of assets and we want to see whether the differences are significant. That means we test  $H_0: \rho_L = \rho_U$  against  $H_1: \rho_L \neq \rho_U$  by using the block-bootstrap procedure described in Section 3.2. The block length corresponds to  $b = 40$  (i.e. approximately 2 trading months) and the number of bootstrap replications is  $N_B = 1000$ . After computing the first estimate  $\hat{\tau}_{LR,b}^2$ , a second run with block length  $b/2 = 20$  is made. This leads to the second estimate  $\hat{\tau}_{LR,b/2}^2$  for the long-run variance and the linear combination

$$\hat{\tau}_{LR}^2 = 2\hat{\tau}_{LR,b}^2 - \hat{\tau}_{LR,b/2}^2$$

is chosen as an estimate for  $\tau_{LR}^2$ . Such a linear combination typically leads to more accurate estimates of the long-run variance than taking a single estimate (Politis, 2003).

Each rejection of a hypothesis test can be interpreted as an outcome of a Bernoulli experiment with parameter value  $0 < \pi_i < 1$ . The considered test statistics indeed depend on each other but nevertheless an unbiased estimate of the *rejection rate*  $\bar{\pi} := 1/435 \sum_{i=1}^{435} \pi_i$  is given by the proportion of rejections. Since the considered tests are unbiased (see Figure 1),  $H_1$  is said to be ‘true in general’ if  $\bar{\pi} \gg \alpha$ . Note that  $\min_i \pi_i \leq \bar{\pi}$ , i.e. the ‘worst’ of the 435 asset combinations cannot produce a power which is larger than  $\bar{\pi}$  and so the rejection rate may serve as an upper bound for the most optimistic view in favor of  $H_1$ . Conversely, given a *one-sided* hypothesis test,  $\bar{\pi} \ll \alpha$  implies that  $H_0$  is ‘true in general’, since the considered hypothesis tests are not conservative but strictly decreasing in  $H_1$  (see Table 4 and Table 5).

The first panel of Table 7 contains the proportions of rejections for all 435 asset combinations. For the shortfall probability  $p = 0.1$  only 10% asset combinations exhibit significantly different Spearman’s rhos on a significance level of  $\alpha = 0.1$ . However, it can be seen that for all  $p \geq 0.2$  the proportions of rejections exceed the corresponding significance levels. Especially, if  $p$  increases the rejection rates apparently become very large and so we conclude that the lower and upper conditional rank-correlation coefficients in general are different from each other.

### 5.2 One-Sided Hypothesis Tests

Panel 2 and 3 of Table 7 contain the proportions of asset combinations where the lower conditional Spearman’s rho exceeds the upper conditional Spearman’s rho and vice versa. For example, 67% of the asset combinations are such that  $\hat{\rho}_{L,n} > \hat{\rho}_{U,n}$  given the shortfall probability  $p = 0.1$  but only 17% of these combinations are significant on the significance level  $\alpha = 0.1$ . It is clear that not every combination with  $\hat{\rho}_{L,n} > \hat{\rho}_{U,n}$  or  $\hat{\rho}_{L,n} < \hat{\rho}_{U,n}$  can be significant. This holds especially if the number of observations in the lower left and upper right area of the empirical copula is small. So even if the proportion of significant combinations might appear to be somewhat

| Panel 1 $H_0: \rho_L = \rho_U$ vs. $H_1: \rho_L \neq \rho_U$ |           |           |           |           |           |  |
|--------------------------------------------------------------|-----------|-----------|-----------|-----------|-----------|--|
| $\alpha$                                                     | $p = 0.1$ | $p = 0.2$ | $p = 0.3$ | $p = 0.4$ | $p = 0.5$ |  |
| 0.10                                                         | .10       | .27       | .46       | .63       | .78       |  |
| 0.05                                                         | .05       | .20       | .37       | .53       | .69       |  |
| 0.01                                                         | .01       | .09       | .24       | .36       | .49       |  |
| Panel 2 $H_0: \rho_L \leq \rho_U$ vs. $H_1: \rho_L > \rho_U$ |           |           |           |           |           |  |
| $\alpha$                                                     | $p = 0.1$ | $p = 0.2$ | $p = 0.3$ | $p = 0.4$ | $p = 0.5$ |  |
| 0.10                                                         | .17       | .37       | .57       | .73       | .84       |  |
| 0.05                                                         | .09       | .25       | .45       | .62       | .78       |  |
| 0.01                                                         | .02       | .12       | .29       | .42       | .57       |  |
| $\hat{\rho}_{L,n} > \hat{\rho}_{U,n}$                        | .67       | .79       | .92       | .95       | .97       |  |
| Panel 3 $H_0: \rho_L \geq \rho_U$ vs. $H_1: \rho_L < \rho_U$ |           |           |           |           |           |  |
| $\alpha$                                                     | $p = 0.1$ | $p = 0.2$ | $p = 0.3$ | $p = 0.4$ | $p = 0.5$ |  |
| 0.10                                                         | .03       | .03       | .01       | .01       | .00       |  |
| 0.05                                                         | .01       | .02       | .00       | .01       | .00       |  |
| 0.01                                                         | .00       | .01       | .00       | .00       | .00       |  |
| $\hat{\rho}_{L,n} < \hat{\rho}_{U,n}$                        | .33       | .21       | .08       | .05       | .03       |  |

**Table 7** Proportions of rejections of the different hypothesis tests, shortfall probabilities, and significance levels for the 435 asset combinations. The proportions of asset combinations where  $\hat{\rho}_L$  is larger or smaller than  $\hat{\rho}_U$  are presented in the last rows of Panel 2 and Panel 3.

small, it neither implies that most of the null hypotheses are true nor that the differences of the lower and upper conditional rank-correlation coefficients are ‘small’ (see the last row of Table 6).

The second panel of Table 7 clearly reveals that the rejection rates of the hypothesis tests for various levels of  $p$  and  $\alpha$  considerably exceed the corresponding significance levels. This effect becomes more obvious the more  $p$  increases. Hence, we have found a strong evidence for the hypothesis  $H_1: \rho_L > \rho_U$ . In contrast, for the opposite test the proportions of rejections given in Table 7 (Panel 3) are substantially smaller than the significance levels. Once again this clearly supports the aforementioned hypothesis.

Many empirical studies suggest that the *linear correlation coefficient* of stock returns is larger in bear markets than in bull markets (see, e.g., Ang and Chen, 2002, Erb et al., 1994). Our results of the one-sided hypothesis tests confirm these findings in the literature, where Pearson’s rho is used as a dependence measure. That means in bear markets daily stock returns depend more on each other than in bull markets. This holds even if ‘dependence’ is measured by Spearman’s rho, which is neither susceptible to outliers nor affected by the marginal distributions of the considered random variables.

## 6 Conclusion

Several authors have investigated the dependencies of stock returns in bull and bear markets. Pearson's rho has been typically used as a canonical dependence measure. Unfortunately, it essentially depends on the marginal cumulative distribution functions of the random variables which are taken into consideration and quantifies only the degree of linear dependence. However, one is often interested in the degree of monotone rather than linear dependence. This holds especially if the marginal distributions are highly non-standard which is definitely the case when concentrating on the tails of stock return distributions. So it is crucial to find a reasonable dependence measure for the degree of monotone dependence under the condition that stock returns contemporaneously go up or down. We believe that copula theory can serve as an appropriate tool-box and suggest Spearman's rho as a dependence measure. This is in contrast to the previous literature, where e.g. conditional versions of Pearson's rho have been used for the same purpose. Moreover, our approach is purely nonparametric. Since we do not fit specific copulas to the data or suggest specific time series models, we are able to avoid any kind of model misspecification. The finite-sample performance of the proposed hypothesis tests have been demonstrated by Monte Carlo simulation. Further, an empirical study using daily returns of stocks contained in the DAX 30 has been conducted. We think that there is sufficient evidence to support the hypothesis of different degrees of monotone dependence in bull and bear markets.

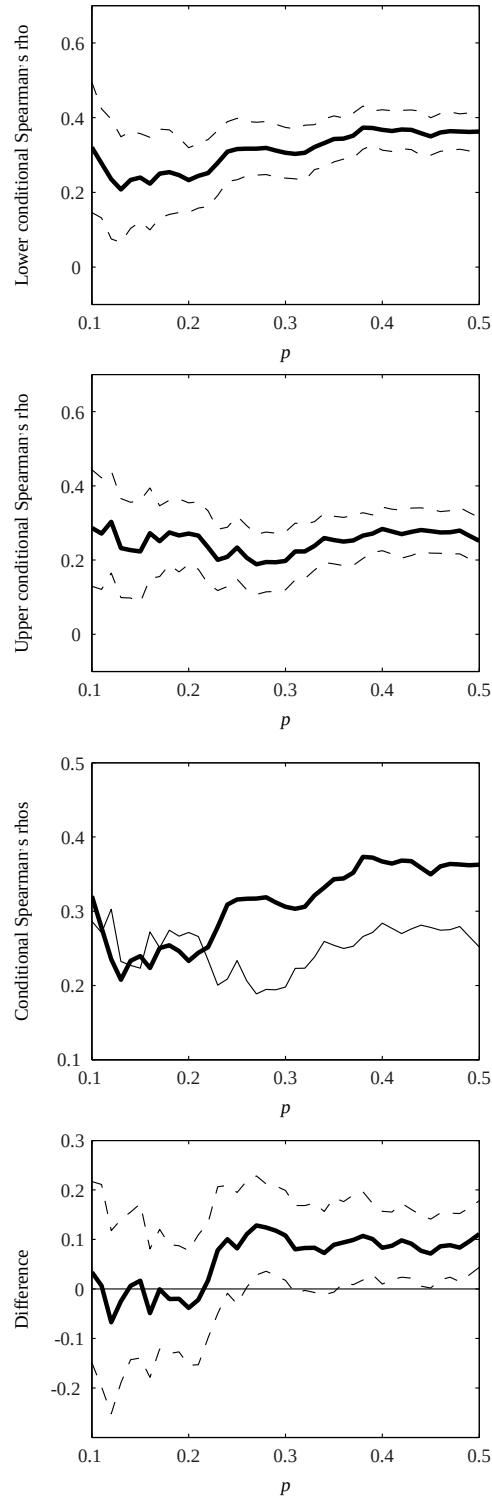
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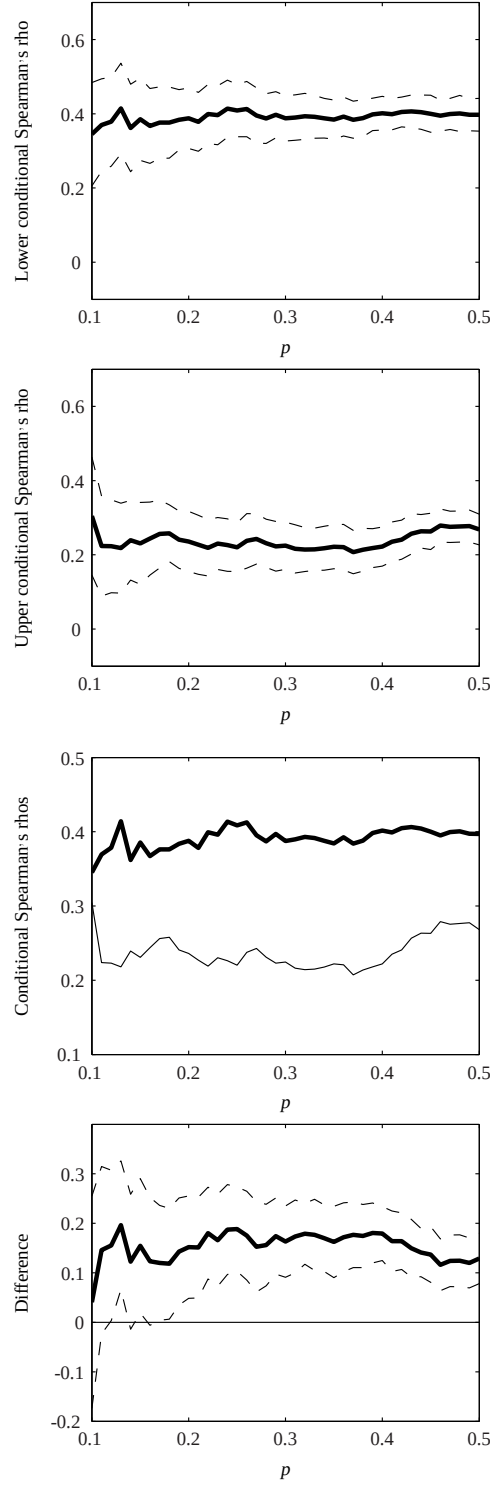
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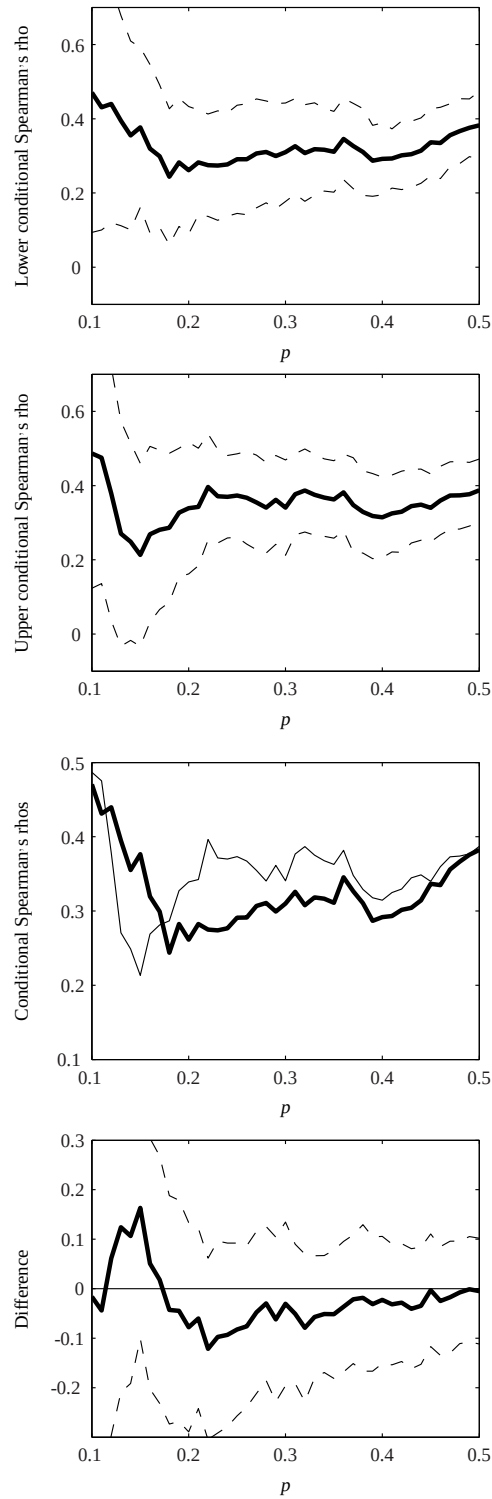
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**Fig. 2** The upper part shows the lower (left hand side) and upper (right hand side) conditional Spearman's rho as a function of  $p = q$  for BASF vs. Henkel ( $n = 5343$ ). The dashed lines represent the corresponding 95%-confidence bands. In the lower left part the lower and upper conditional Spearman's rhos are shown together where the thick line represents the lower conditional Spearman's rho. The lower right part contains the difference between the rhos and the corresponding 95%-confidence band.



**Fig. 3** The upper part shows the lower (left hand side) and upper (right hand side) conditional Spearman's rho as a function of  $p = q$  for BASF vs. Thyssen ( $n = 7884$ ). The dashed lines represent the corresponding 95%-confidence bands. In the lower left part the lower and upper conditional Spearman's rhos are shown together where the thick line represents the lower conditional Spearman's rho. The lower right part contains the difference between the rhos and the corresponding 95%-confidence band.



**Fig. 4** The upper part shows the lower (left hand side) and upper (right hand side) conditional Spearman's rho as a function of  $p = q$  for BASF vs. Infineon ( $n = 2110$ ). The dashed lines represent the corresponding 95%-confidence bands. In the lower left part the lower and upper conditional Spearman's rhos are shown together where the thick line represents the lower conditional Spearman's rho. The lower right part contains the difference between the rhos and the corresponding 95%-confidence band.