

Poverty Measures in a Fuzzy Logic Framework

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Abstract In the income (or consumption) poverty framework, a relevant open problem is connected with the distinction between poor and non-poor. This is usually done by considering a pre-specified income threshold called poverty line. If the income of an observation unit is lower (higher) than the poverty line, then such an observation unit is poor (non-poor or rich). Unfortunately, this procedure may often lead to questionable results because the concept of poverty is vague. In this connection, a promising line of research is the measurement of poverty according to a fuzzy logic approach. In this work, following such an approach, some new poverty measures based on the Gini and Bonferroni inequality indices are proposed.

Keywords Measuring poverty and inequality, Fuzzy logic approach, Sen poverty index

1 Introduction

In the last few decades, a great deal of attention has been paid to the analysis and measurement of poverty in order to understand the phenomenon and make the policy makers able to defeat it. Poverty measurement is mainly evaluated in terms of monetary variables (e.g. income, consumption).

Let X be the distribution of a monetary variable concerning a set of n observation units. For instance, without loss of generality, we suppose that X denotes an income distribution. Specifically, we indicate by x_i the income of the i -th observation unit ($i=1, \dots, n$) and we assume that $0 \leq x_1 \leq x_2 \leq \dots \leq x_n$. Our primary interest relies in assessing whether a given observation unit can be considered poor or not. The classical approach to poverty mainly consists of introducing a poverty line, say x_P , and concluding that the generic i -th observation unit is poor when $x_i < x_P$ (and non-poor or rich otherwise).

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Unfortunately, this approach can often be limited. For example, suppose to deal with three observation units (x_1 , x_2 and x_3) such that $x_1 = x_P - \varepsilon$, $x_2 = x_P + \varepsilon$ (with $\varepsilon > 0$) and $x_3 \gg x_P$. Following the classical approach to poverty, we conclude that the first observation is poor, whereas the remaining two are non-poor. However, this contradicts the common human thinking. Let us start with observations 1 and 2. From a mathematical point of view, as $x_1 = x_P - \varepsilon$ and $x_2 = x_P + \varepsilon$, the former is considered poor and the latter non-poor. Conversely, from a practical point of view, both observations can be evaluated as poor. In fact, also observation 2 is *approximately* poor because her/his income is very close to the poverty line, even if slightly higher than. Also the status of observations 2 and 3 is in contrast with the human common-sense reasoning. In fact, both observations are considered non-poor, whereas it seems to be more appropriate to state that the former is non-poor *to some very limited extent* and the latter is *definitively* non-poor. Conversely, to avoid this drawback, fuzzy logic [12] seems to be a more promising tool of research. The fuzzy logic theory is a precise theory for dealing with imprecision and vagueness. Generally speaking, the vagueness associated to the concept of poverty can be highlighted with respect to at least two characteristics. First, it is not reasonable that, when the income of an observation increases by a very small amount, such an observation moves from the poor status to the non-poor one. Furthermore, it would be desirable to classify some observations as borderline poor (or poor to some extent) in order to suitably distinguish different levels of poverty.

2 Fuzzy logic approach to poverty

In a fuzzy logic setting we assume that there exist a poverty line x_P and a richness line x_R ($x_P < x_R$). The generic i -th observation is considered poor (non-rich) if $x_i < x_P$. In a similar way, we say that the i -th observation is rich (non-poor) when $x_i \geq x_R$. Finally, when $x_P \leq x_i < x_R$, we can conclude that the i -th observation belongs to a limbo corresponding to the non-poor & non-rich case. Hence, since the i -th observation cannot be considered rich, it follows that such an observation is poor *to some extent*. Therefore, we no longer distinguish poor and rich (non-poor) observation units in a dichotomous way. Rather, following the fuzzy logic approach, we assign to every observation unit a degree of poverty ranging from 0 to 1 according to the corresponding income. This can be done by introducing the so-called membership function (of an observation i to the attribute ‘poor’). In particular, if $\tilde{X}$ indicates the attribute ‘poor’ (the symbol ‘ $\sim$ ’ denotes that a fuzzy logic approach is adopted), the membership function $\mu_{\tilde{X}} : \mathbb{R}^+ \rightarrow [0,1]$ allows us to express to which extent the i -th observation is poor. This is done by quantifying the degree according to which x_i belongs to $\tilde{X}$. The membership function can be defined as

$$\mu_{\tilde{X}}(x_i) = \begin{cases} 1 & x_i < x_P, \\ f(x_i) & x_P \leq x_i < x_R, \\ 0 & x_i \geq x_R, \end{cases} \quad (1)$$

where $f(x_i)$ is a decreasing function from $[x_p, x_R)$ to $[0,1]$ such that $f(x_p)=1, f(x_i)<1 \forall x_i>x_p, f(x_i)>0 \forall x_i<x_R$, and $\lim_{x_i \rightarrow x_R} f(x_i) = 0$. A possible choice for f is

$$f(x_i) = 1 - \left(\frac{x_i - x_p}{x_R - x_p} \right)^\beta, \text{ where } \beta \text{ is a positive parameter tuning the decreasing trend of}$$

f . In particular, when $\beta = 1$ such a trend is linear. Note that, in this case, the membership function in (1) coincides with that proposed by Cerioli and Zani [2].

It is worth mentioning that rewriting the concept of poverty in terms of membership function according to the classical approach leads to

$$\mu_X(x_i) = \begin{cases} 1 & x_i < x_p, \\ 0 & x_i \geq x_p, \end{cases} \quad (2)$$

from which it should be clear that an observation unit can be considered *strictly* either poor or non-poor (tertium non datur).

For a more deeper insight into the fuzzy logic approach to poverty, we refer to Lemmi and Betti [9], which contains some interesting papers and an exhaustive literature review of the topic. In particular, one can see Qizilbash [10] for a philosophical justification of the fuzzy logic approach and Chakravarty [3] for a rigorous discussion on desirable axioms for poverty indices in a fuzzy logic environment.

3 A review of two poverty measures following the classical approach

In the literature, there is a wide range of works devoted to the development of measures for evaluating the deprivation level of a population. For a detailed review one can refer to [4]. In this paper, we focus our attention on the poverty measures proposed by Sen [11] and Giorgi and Crescenzi [6]. The former is based on the Gini index, whereas the latter on the Bonferroni one. In order to recall the above mentioned measures, it is fruitful to define the following “ingredients” to be used in their set-up. First of all, let

$$p = \sum_{i=1}^n U\{x_i < x_p\} \text{ be the number of poor, where } U \text{ denotes the indicator function.}$$

With regard to the poor observation units, the poverty-gap associated to the i -th observation is $g_i = x_p - x_i$. Thus, g_i is the (positive) difference between the poverty

threshold x_p and the income x_i ($i=1, \dots, p$). Let $\bar{g} = (1/p) \sum_{i=1}^p g_i$ be the mean of the

poverty-gaps of the poor. The income mean over the p poor observation units is

$$m_p = (1/p) \sum_{i=1}^p x_i \text{ and the income mean over the } i \text{ poorest observation units } (i=1, \dots, p-1)$$

is $m_i = (1/i) \sum_{j=1}^i x_j$. We can now define the following three ratios, which will be used

for determining the Sen poverty measure:

$$I = \frac{\sum_{i=1}^p g_i}{x_p p} = \frac{\bar{g}}{x_p}, \quad (3)$$

$$G_p = \frac{p-1}{p} R_p = \frac{p-1}{p} \left(1 - \frac{2 \sum_{i=1}^{p-1} i m_i}{p(p-1) m_p} \right), \quad (4)$$

$$H = \frac{p}{n}. \quad (5)$$

The ratio in (3) is usually known as the poverty-gap ratio of the poor. The index in (4) is the Gini coefficient computed among the poor. As it is well-known, it represents a measure of inequality. Note that, up to the constant $(p-1)/p$, the Gini coefficient among the poor is equal to the Gini concentration index among the poor R_p , implicitly defined in (4). Finally, (5) is the head-count ratio.

By taking into account the ratios in (3)-(5), Sen [11] suggests the following poverty measure:

$$S = H \left\{ 1 - (1 - I) \left[1 - \frac{p}{p+1} G_p \right] \right\}. \quad (6)$$

It takes values in $[0,1]$. S is equal to 0 when there are no poor and equal to 1 when all the observation units have no income ($x_1 = \dots = x_n = 0$). It is important to note that Sen [11] proposes (6) according to an axiomatic framework.

A modification of S has been proposed by Giorgi and Crescenzi [6], who suggest to use the Bonferroni index among the poor in place of the Gini one. The Bonferroni index among the poor is defined as

$$B_p = \frac{1}{p-1} \sum_{i=1}^{p-1} \frac{m_p - m_i}{m_p} = 1 - \frac{1}{p-1} \sum_{i=1}^{p-1} \frac{m_i}{m_p}, \quad (7)$$

where m denotes the income mean computed over the entire distribution.

It is worth mentioning that, from a theoretical point of view, poverty measures based on B_p are more powerful than those based on G_p in order to inspect an income distribution. For a comparative assessment of the Gini and Bonferroni indices see [6].

In an axiomatic framework, the Giorgi and Crescenzi [6] measure is given by

$$S_b = H \left\{ 1 - (1 - I) \left[1 - \frac{p-1}{p} B_p \right] \right\}. \quad (8)$$

By comparing (6) and (8), it is easy to see that the main difference relies in the use of G_p (weighted by $p/(p+1)$) in S and that of B_p (weighted by $(p-1)/p$) in S_b .

4 Some proposals of poverty measures following the fuzzy logic approach

In this section, we extend in a fuzzy logic approach the measures recalled in (6) and (8). This is done by generalizing the above poverty measures making use of the fuzzy logic definition of poverty given in (1). From a practical point of view, this implies the development of suitable extensions of the ratios and indices in (3)-(5) and (7).

The first step towards the fuzzy logic generalization of (6) and (8) involves a different way to assess whether the observations are poor. In fact, this should be done considering not only the (completely) poor observations (membership function = 1) but also the poor to some extent (membership function in (0,1)), already labelled as non-poor & non-rich. As a consequence, bearing in mind that the membership function of a rich is zero, the evaluation of the number of poor p can be written as

$$\tilde{p} = \sum_{i=1}^n \mu_{\tilde{x}}(x_i), \quad (9)$$

where the symbol ‘tilde’ is used in order to highlight that the fuzzy logic approach is followed¹. Moreover, we denote by r and $\bar{r}$ the numbers of rich ($r = \sum_{i=1}^n U\{x_i \geq x_R\}$)

and non-rich ($\bar{r} = \sum_{i=1}^n U\{x_i < x_R\}$), respectively. Note that $\bar{r} \geq p$.

Following the fuzzy logic approach, the poverty-gap ratio in (3) can be extended as

$$\tilde{I} = \frac{\sum_{i=1}^{\bar{r}} \tilde{g}_i \mu_{\tilde{x}}(x_i)}{x_R \tilde{p}} = \frac{\tilde{g}}{x_R}, \quad (10)$$

with $\tilde{g}_i = x_R - x_i$. Thus, the gap is no longer computed with respect to the poverty line but with respect to the richness line. Therefore, it appears to be more suitable to refer to (10) as the non-richness-gap ratio of the non-rich.

The fuzzy logic extensions of G_p and B_p can be developed by modifying the quantities m_p and m_i 's. First of all, in a fuzzy logic context, attention should be paid to the $\bar{r}$ non-rich (not only to the p poor). Moreover, these average values can be computed considering the system of weights given by the membership function

information. We have, respectively, $\tilde{m}_{\bar{r}} = \frac{\sum_{i=1}^{\bar{r}} x_i \mu_{\tilde{x}}(x_i)}{\sum_{i=1}^{\bar{r}} \mu_{\tilde{x}}(x_i)}$ and $\tilde{m}_i = \frac{\sum_{j=1}^i x_j \mu_{\tilde{x}}(x_j)}{\sum_{j=1}^i \mu_{\tilde{x}}(x_j)}$,

$i=1, \dots, \bar{r}-1$. We then get

¹ It is important to remark that the symbol ‘tilde’ is usually adopted for denoting a fuzzy set. However, in (9) as well as in the sequel, this symbol will be used for denoting (non-fuzzy) quantities based on the fuzzy logic approach to poverty.

$$\tilde{G}_{\bar{r}} = \frac{\bar{r}-1}{\bar{r}} \tilde{R}_{\bar{r}} = \frac{\bar{r}-1}{\bar{r}} \left(1 - \frac{2 \sum_{i=1}^{\bar{r}-1} i \tilde{m}_i}{\bar{r}(\bar{r}-1) \tilde{m}_{\bar{r}}} \right), \quad (11)$$

$$\tilde{B}_{\bar{r}} = \frac{1}{\bar{r}-1} \sum_{i=1}^{\bar{r}-1} \frac{\tilde{m}_{\bar{r}} - \tilde{m}_i}{\tilde{m}_{\bar{r}}} = 1 - \frac{1}{\bar{r}-1} \sum_{i=1}^{\bar{r}-1} \frac{\tilde{m}_i}{\tilde{m}_{\bar{r}}}. \quad (12)$$

Finally, in a fuzzy logic setting, the Head Count ratio in (5) can be generalized as

$$\tilde{H} = \frac{\sum_{i=1}^n \mu_{\tilde{X}}(x_i)}{n} = \frac{\tilde{p}}{n}. \quad (13)$$

Taking into account (10)-(13), it can be shown that the fuzzy logic generalizations of (6) and (8) are, respectively,

$$\tilde{S} = \tilde{H} \left\{ 1 - (1 - \tilde{I}) \left[1 - \frac{\bar{r}}{\bar{r}+1} \tilde{G}_{\bar{r}} \right] \right\}, \quad (14)$$

$$\tilde{S}_B = \tilde{H} \left\{ 1 - (1 - \tilde{I}) \left[1 - \frac{\bar{r}-1}{\bar{r}} \tilde{B}_{\bar{r}} \right] \right\}. \quad (15)$$

See, for more details, Giordani and Giorgi [5].

5 Application

In this section we determine the values of $\tilde{S}$ and $\tilde{S}_B$ for a real data set. The data here considered come from the 2001 Istat (Italian national institute of statistics) Household Budget Survey, abbreviated as HBS. Specifically, to measure poverty in Italy, Istat uses a survey based on a sample of private households. A two-stage cluster sample is adopted where the “primary” sampling units are municipalities and the “secondary” sampling units are households. A stratified sampling is used for the “primary” sampling units. The main purpose of the survey is to give a picture of living conditions of private households mainly focusing on their consumption expenditure, broken down by household characteristics such as income, socio-economic characteristics, size and composition, housing conditions and so forth. The expenditure effected by households to acquire goods and services is recorded allowing us to determine the household total monthly consumption expenditure (obtained by means of a suitable Istat data processing). The total monthly consumption expenditure (net from expenses on extraordinary home maintenance, loans and premiums paid for life insurance and life annuity that are not part of the economic concept of consumption expenses) is the main variable used for analyzing poverty. In fact, to assess whether a given household is

poor, Istat compares the corresponding total monthly consumption expenditure with a (monthly) poverty threshold to be discussed below. Finally, a weight is associated to every household in order to obtain suitable estimates of the population statistics. See, for further details, [7].

To compute $\tilde{S}$ and $\tilde{S}_B$ the following steps are needed. First, we determine the total monthly consumption expenditure for every household. However, this is not yet the distribution to be analyzed for deriving the values of $\tilde{S}$ and $\tilde{S}_B$. In fact, the standard poverty measure (hereinafter, *spm*) adopted by Istat (following the International Standard of Poverty Line) is equal to the mean per capita total monthly consumption expenditure (€ 814.55 in 2001). Such a poverty threshold makes sense only for households with two components. Thus, it must be modified for households with number of members different from two according to the Carbonaro equivalence scale [1]. This allows us to assess poverty for different sized households. However, for our purpose, we need to perform the inverse rescaling procedure. More specifically, let nf be the normalizing factor (for instance $nf = 1.33$ for a household with three components). Rather than normalizing the poverty threshold *spm* by means of the normalizing factor nf (and comparing it with the observed total monthly consumption expenditure), we need to normalize every total monthly consumption expenditure. This is done by multiplying it by $1/nf$ according to the household size. For instance, if the observed total monthly consumption expenditure of a household with three components is € 1,500, then the normalized one is $1,500 / 1.33 = 1,127.82$.

The next step is how to choose the poverty and richness thresholds. To do it, we decide to follow some suggestions by Istat [8]. A household is considered *definitively* poor if its total monthly consumption expenditure is lower than a percentage of 80% of *spm* ($0.8 \times 814.55 = 651.64$). We thus set $x_P = € 651.64$. Similarly, a household is considered *definitively* non-poor (i.e. *definitively* rich) if its total monthly consumption expenditure is higher than a percentage of 120% of *spm* ($1.2 \times 814.55 = 977.46$). In this way we obtain $x_R = € 977.46$. Note also that Istat refers to households with total monthly consumption expenditure between € 651.64 (80% of *spm*) and € 814.55 (100% of *spm*) as *nearly* poor and those with total monthly consumption expenditure between € 814.55 (100% of *spm*) and € 977.46 (120% of *spm*) as *almost* poor. Therefore, we can say that Istat implicitly considers the concept of poverty in a vague way (admitting degrees of poverty) without, unfortunately, managing it by a fuzzy logic approach. In fact, a household considered almost poor would be poor to some extent (i.e. with a membership function value strictly lower than one). Nonetheless, this is fully missed by Istat which introduces some classical (non-fuzzy) sets (definitively poor, nearly poor, almost poor, definitively non-poor) according to the classical approach to poverty rather than a unique fuzzy set allowing us to deal jointly with different degrees of poverty.

The last step concerns the way to take into account the weights associated to the households, say α_i 's. In order to estimate the population values of $\tilde{S}$ and $\tilde{S}_B$, suitable modifications of (9)-(13) involving the use of the weights α_i 's. are required. These are denoted as $p_\alpha, \tilde{I}_\alpha, \tilde{G}_{\bar{\alpha}}, \tilde{B}_{\bar{\alpha}}$ and $\tilde{H}_\alpha$, respectively. This allows us to get

$$\tilde{S}_\alpha = \tilde{H}_\alpha \left\{ 1 - (1 - \tilde{I}_\alpha) \left[1 - \frac{\sum_{j=1}^{\tilde{r}} \alpha_j}{\sum_{j=1}^{\tilde{r}+1} \alpha_j} \tilde{G}_{\tilde{R}\alpha} \right] \right\}, \quad (16)$$

$$\tilde{S}_{B\alpha} = \tilde{H}_\alpha \left\{ 1 - (1 - \tilde{I}_\alpha) \left[1 - \frac{\sum_{j=1}^{\tilde{r}-1} \alpha_j}{\sum_{j=1}^{\tilde{r}} \alpha_j} B_{\tilde{R}\alpha} \right] \right\}, \quad (17)$$

and, setting $\beta=1$, we find $\tilde{S}_\alpha = 0.1322$ and $\tilde{S}_{B\alpha} = 0.0572$. For further details, see [5].

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