

# On Gaussian Compound Poisson Type Limiting Likelihood Ratio Process

Sergueï Dachian and Ilia Negri

**Abstract** Different change-point type models encountered in statistical inference for stochastic processes give rise to different limiting likelihood ratio processes. Recently it was established that one of these likelihood ratios, which is an exponential functional of a two-sided Poisson process driven by some parameter, can be approximated (for sufficiently small values of the parameter) by another one, which is an exponential functional of a two-sided Brownian motion. In this paper we consider yet another likelihood ratio, which is the exponent of a two-sided compound Poisson process driven by some parameter. We establish that the compound Poisson type likelihood ratio can also be approximated by the Brownian type one for sufficiently small values of the parameter. We equally discuss the asymptotics for large values of the parameter.

**Key words:** probability and stochastic processes, parametric estimation, change-point, non-regularity, Bayesian estimators, maximum likelihood estimator, limiting distribution, limiting variance, asymptotic efficiency

## 1 Introduction

Different change-point type models encountered in statistical inference for stochastic processes give rise to different limiting likelihood ratio processes. In [3] a relation between two of these likelihood ratios was established by one of the authors. More precisely, it was shown that the first one, which is an exponential functional

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of a two-sided Poisson process driven by some parameter, can be approximated (for sufficiently small values of the parameter) by the second one, defined by

$$Z_0(x) = \exp \left\{ W(x) - \frac{1}{2} |x| \right\}, \quad x \in \mathbb{R}, \quad (1)$$

where  $W$  is a standard two-sided Brownian motion. In this paper we consider another limiting likelihood ratio process arising in some change-point type models.

We introduce the random process  $Z_\gamma$  on  $\mathbb{R}$  as the exponent of a two-sided compound Poisson process given by

$$\ln Z_\gamma(x) = \begin{cases} \gamma \sum_{k=1}^{\Pi_+(x)} \varepsilon_k^+ - \frac{\gamma^2}{2} \Pi_+(x), & \text{if } x \geq 0, \\ \gamma \sum_{k=1}^{\Pi_-(-x)} \varepsilon_k^- - \frac{\gamma^2}{2} \Pi_-(-x), & \text{if } x \leq 0, \end{cases} \quad (2)$$

where  $\gamma > 0$ ,  $\Pi_+$  and  $\Pi_-$  are two independent Poisson processes of intensity 1 on  $\mathbb{R}_+$ ,  $\varepsilon_k^\pm$  are independent standard Gaussian random variables which are also independent of  $\Pi_\pm$ , and we use the convention  $\sum_{k=1}^0 \varepsilon_k^\pm = 0$ . We equally introduce the random variables

$$\begin{aligned} \zeta_\gamma &= \frac{\int_{\mathbb{R}} x Z_\gamma(x) dx}{\int_{\mathbb{R}} Z_\gamma(x) dx}, \\ \xi_\gamma^- &= \inf \left\{ z : Z_\gamma(z) = \sup_{x \in \mathbb{R}} Z_\gamma(x) \right\}, \\ \xi_\gamma^+ &= \sup \left\{ z : Z_\gamma(z) = \sup_{x \in \mathbb{R}} Z_\gamma(x) \right\}, \\ \xi_\gamma^\alpha &= \alpha \xi_\gamma^- + (1 - \alpha) \xi_\gamma^+, \quad \alpha \in [0, 1], \end{aligned} \quad (3)$$

related to this process, as well as their second moments  $B_\gamma = \mathbf{E} \zeta_\gamma^2$  and  $M_\gamma^\alpha = \mathbf{E} (\xi_\gamma^\alpha)^2$ .

The process  $Z_\gamma$ , up to a linear time change, arises in some non-regular, namely change-point type, statistical models (for example, the threshold autoregressive model studied by Chan and Kutoyants in [21]) as the limiting likelihood ratio process, and the variables  $\zeta_\gamma$  and  $\xi_\gamma^\alpha$  as the limiting distributions of the Bayesian estimators and of the appropriately chosen maximum likelihood estimator, respectively. Here the maximum likelihood estimator is not unique, and the appropriate choice is a linear combination with weights  $\alpha$  and  $1 - \alpha$  of its minimal and maximal values. In particular,  $B_\gamma$  and  $M_\gamma^\alpha$  are the limiting variances of these estimators, and the Bayesian estimators being asymptotically efficient, the ratio  $E_\gamma^\alpha = B_\gamma / M_\gamma^\alpha$  is the asymptotic efficiency of this maximum likelihood estimator.

On the other hand, many change-point type statistical models encountered in various fields of statistical inference for stochastic processes rather have as limiting likelihood ratio process, up to a linear time change, the process  $Z_0$ . In this case, the limiting distributions of the Bayesian estimators and of the maximum likelihood

estimator are given by

$$\zeta_0 = \frac{\int_{\mathbb{R}} x Z_0(x) dx}{\int_{\mathbb{R}} Z_0(x) dx} \quad \text{and} \quad \xi_0 = \operatorname{argsup}_{x \in \mathbb{R}} Z_0(x), \quad (4)$$

respectively, and the limiting variances of these estimators are  $B_0 = \mathbf{E}\zeta_0^2$  and  $M_0 = \mathbf{E}\xi_0^2$ .

A well-known example is the model of a discontinuous signal in a white Gaussian noise exhaustively studied by Ibragimov and Khasminskii in [11, Chapter 7.2] (see also their previous work [10]), but one can also cite change-point type models of dynamical systems with small noise considered by Kutoyants in [13] and [14, Chapter 5], those of ergodic diffusion processes examined by Kutoyants in [15, Chapter 3], a change-point type model of delay equations analyzed by Küchler and Kutoyants in [12], an i.i.d. change-point type model explored by Deshayes and Picard in [5], a model of a discontinuous periodic signal in a time inhomogeneous diffusion investigated by Höpfner and Kutoyants in [9], and so on.

Let us also note that Terent'yev in [17] determined explicitly the distribution of  $\xi_0$  and calculated the constant  $M_0 = 26$ . These results were taken up by Ibragimov and Khasminskii in [11, Chapter 7.3], where by means of numerical simulation they equally showed that  $B_0 = 19.5 \pm 0.5$ , and so  $E_0 = 0.73 \pm 0.03$ . Later in [8], Golubev expressed  $B_0$  in terms of the second derivative (with respect to a parameter) of an improper integral of a composite function of modified Hankel and Bessel functions. Finally in [16], Rubin and Song obtained the exact values  $B_0 = 16\zeta(3)$  and  $E_0 = 8\zeta(3)/13$ , where  $\zeta$  is Riemann's zeta function defined by  $\zeta(s) = \sum_{n=1}^{\infty} 1/n^s$ .

In this paper we establish that the limiting likelihood ratio processes  $Z_\gamma$  and  $Z_0$  are related. More precisely, we show that as  $\gamma \rightarrow 0$ , the process  $Z_\gamma(y/\gamma^2)$ ,  $y \in \mathbb{R}$ , converges weakly in the space  $\mathcal{D}_0(-\infty, +\infty)$  (the Skorohod space of functions on  $\mathbb{R}$  without discontinuities of the second kind and vanishing at infinity) to the process  $Z_0$ . So, the random variables  $\gamma^2 \zeta_\gamma$  and  $\gamma^2 \xi_\gamma^\alpha$  converge weakly to the random variables  $\zeta_0$  and  $\xi_0$ , respectively. We show equally that the convergence of moments of these random variables holds, that is,  $\gamma^4 B_\gamma \rightarrow 16\zeta(3)$ ,  $\gamma^4 M_\gamma^\alpha \rightarrow 26$  and  $E_\gamma^\alpha \rightarrow 8\zeta(3)/13$ .

These are the main results of the present paper, and they are presented in Section 2, where we also briefly discuss the second possible asymptotics  $\gamma \rightarrow +\infty$ . The necessary lemmas are proved in Section 3 and, finally, in Section 4 we discuss some directions for future development.

## 2 Asymptotics of the limiting likelihood ratio

Consider the process  $X_\gamma(y) = Z_\gamma(y/\gamma^2)$ ,  $y \in \mathbb{R}$ , where  $\gamma > 0$  and  $Z_\gamma$  is defined by (2). Note that

$$\frac{\int_{\mathbb{R}} y X_{\gamma}(y) dy}{\int_{\mathbb{R}} X_{\gamma}(y) dy} = \gamma^2 \zeta_{\gamma},$$

$$\inf \left\{ z : X_{\gamma}(z) = \sup_{y \in \mathbb{R}} X_{\gamma}(y) \right\} = \gamma^2 \xi_{\gamma}^{-}$$

and

$$\sup \left\{ z : X_{\gamma}(z) = \sup_{y \in \mathbb{R}} X_{\gamma}(y) \right\} = \gamma^2 \xi_{\gamma}^{+},$$

where the random variables  $\zeta_{\gamma}$  and  $\xi_{\gamma}^{\pm}$  are defined by (3). Remind also the process  $Z_0$  on  $\mathbb{R}$  defined by (1) and the random variables  $\zeta_0$  and  $\xi_0$  defined by (4). Recall finally the quantities  $B_{\gamma} = \mathbf{E} \zeta_{\gamma}^2$ ,  $M_{\gamma}^{\alpha} = \mathbf{E}(\xi_{\gamma}^{\alpha})^2$ ,  $E_{\gamma}^{\alpha} = B_{\gamma}/M_{\gamma}^{\alpha}$ ,  $B_0 = \mathbf{E} \zeta_0^2 = 16 \zeta(3)$ ,  $M_0 = \mathbf{E} \xi_0^2 = 26$  and  $E_0 = B_0/M_0 = 8 \zeta(3)/13$ . Now we can state the main result of the present paper.

**Theorem 1.** *The process  $X_{\gamma}$  converges weakly in the space  $\mathcal{D}_0(-\infty, +\infty)$  to the process  $Z_0$  as  $\gamma \rightarrow 0$ . In particular, the random variable  $\gamma^2 \zeta_{\gamma}$  converge weakly to the random variable  $\zeta_0$  and, for any  $\alpha \in [0, 1]$ , the random variable  $\gamma^2 \xi_{\gamma}^{\alpha}$  converge weakly to the random variable  $\xi_0$ . Moreover, for any  $k > 0$  we have*

$$\gamma^{2k} \mathbf{E} \zeta_{\gamma}^k \rightarrow \mathbf{E} \zeta_0^k \quad \text{and} \quad \gamma^{2k} \mathbf{E} (\xi_{\gamma}^{\alpha})^k \rightarrow \mathbf{E} \xi_0^k.$$

In particular,  $\gamma^4 B_{\gamma} \rightarrow 16 \zeta(3)$ ,  $\gamma^4 M_{\gamma}^{\alpha} \rightarrow 26$  and  $E_{\gamma}^{\alpha} \rightarrow 8 \zeta(3)/13$ .

The results concerning the random variable  $\zeta_{\gamma}$  are direct consequence of [11, Theorem 1.10.2] and the following three lemmas.

**Lemma 1.** *The finite-dimensional distributions of the process  $X_{\gamma}$  converge to those of  $Z_0$  as  $\gamma \rightarrow 0$ .*

**Lemma 2.** *For all  $\gamma > 0$  and all  $y_1, y_2 \in \mathbb{R}$  we have*

$$\mathbf{E} \left| X_{\gamma}^{1/2}(y_1) - X_{\gamma}^{1/2}(y_2) \right|^2 \leq \frac{1}{4} |y_1 - y_2|.$$

**Lemma 3.** *For any  $c \in ]0, 1/8[$  we have*

$$\mathbf{E} X_{\gamma}^{1/2}(y) \leq \exp(-c |y|)$$

for all sufficiently small  $\gamma$  and all  $y \in \mathbb{R}$ .

Note that these lemmas are not sufficient to establish the weak convergence of the process  $X_{\gamma}$  in the space  $\mathcal{D}_0(-\infty, +\infty)$  and the results concerning the random variable  $\xi_{\gamma}^{\alpha}$ . However, the increments of the process  $\ln X_{\gamma}$  being independent, the convergence of its restrictions (and hence of those of  $X_{\gamma}$ ) on finite intervals  $[A, B] \subset \mathbb{R}$  (that is, convergence in the Skorohod space  $\mathcal{D}[A, B]$  of functions on  $[A, B]$  without discontinuities of the second kind) follows from [7, Theorem 6.5.5], Lemma 1 and the following lemma.

**Lemma 4.** *For any  $\varepsilon > 0$  we have*

$$\lim_{h \rightarrow 0} \lim_{\gamma \rightarrow 0} \sup_{|y_1 - y_2| < h} \mathbf{P} \left\{ |\ln X_\gamma(y_1) - \ln X_\gamma(y_2)| > \varepsilon \right\} = 0.$$

Now, Theorem 1 follows from the following estimate on the tails of the process  $X_\gamma$  by standard argument (see, for example, [11]).

**Lemma 5.** *For any  $b \in ]0, 1/12[$  we have*

$$\mathbf{P} \left\{ \sup_{|y| > A} X_\gamma(y) > e^{-bA} \right\} \leq 4e^{-bA}$$

for all sufficiently small  $\gamma$  and all  $A > 0$ .

Before giving the prove of the above lemmas, let us discuss the second possible asymptotics  $\gamma \rightarrow +\infty$ . One can show that in this case, the process  $Z_\gamma$  converges weakly in the space  $\mathcal{D}_0(-\infty, +\infty)$  to the process  $Z_\infty(x) = \mathbb{1}_{\{-\eta < x < \tau\}}$ ,  $x \in \mathbb{R}$ , where  $\eta$  and  $\tau$  are two independent exponential random variables with parameter 1. So, the random variables  $\zeta_\gamma$ ,  $\xi_\gamma^-$ ,  $\xi_\gamma^+$  and  $\xi_\gamma^\alpha$  converge weakly to the random variables

$$\begin{aligned} \zeta_\infty &= \frac{\int_{\mathbb{R}} x Z_\infty(x) dx}{\int_{\mathbb{R}} Z_\infty(x) dx} = \frac{\tau - \eta}{2}, \\ \xi_\infty^- &= \inf \left\{ z : Z_\infty(z) = \sup_{x \in \mathbb{R}} Z_\infty(x) \right\} = -\eta, \\ \xi_\infty^+ &= \sup \left\{ z : Z_\infty(z) = \sup_{x \in \mathbb{R}} Z_\infty(x) \right\} = \tau \end{aligned}$$

and

$$\xi_\infty^\alpha = \alpha \xi_\infty^- + (1 - \alpha) \xi_\infty^+ = (1 - \alpha) \tau - \alpha \eta,$$

respectively. One can equally show that, moreover, for any  $k > 0$  we have

$$\mathbf{E} \zeta_\gamma^k \rightarrow \mathbf{E} \zeta_\infty^k \quad \text{and} \quad \mathbf{E} (\xi_\gamma^\alpha)^k \rightarrow \mathbf{E} (\xi_\infty^\alpha)^k.$$

In particular, denoting  $B_\infty = \mathbf{E} \zeta_\infty^2$ ,  $M_\infty^\alpha = \mathbf{E} (\xi_\infty^\alpha)^2$  and  $E_\infty^\alpha = B_\infty / M_\infty^\alpha$ , we finally have

$$\begin{aligned} B_\gamma &\rightarrow B_\infty = \mathbf{E} \left( \frac{\tau - \eta}{2} \right)^2 = \frac{1}{2}, \\ M_\gamma^\alpha &\rightarrow M_\infty^\alpha = \mathbf{E} ((1 - \alpha) \tau - \alpha \eta)^2 = 6 \left( \alpha - \frac{1}{2} \right)^2 + \frac{1}{2} \end{aligned} \tag{5}$$

and

$$E_\gamma \rightarrow E_\infty^\alpha = \frac{1}{12\left(\alpha - \frac{1}{2}\right)^2 + 1}. \quad (6)$$

Let us note that these convergences are natural, since the process  $Z_\infty$  can be considered as a particular case of the process  $Z_\gamma$  with  $\gamma = +\infty$  if one admits the convention  $+\infty \cdot 0 = 0$ .

Note also, that  $Z_\infty$  is the limiting likelihood ratio process in the problem of estimating the parameter  $\theta$  by i.i.d. uniform observations on  $[\theta, \theta + 1]$ . So, in this problem, the variables  $\zeta_\infty$  and  $\xi_\infty^\alpha$  are the limiting distributions of the Bayesian estimators and of the maximum likelihood estimator, respectively,  $B_\infty$  and  $M_\infty^\alpha$  are the limiting variances of these estimators and, the Bayesian estimators being asymptotically efficient,  $E_\infty^\alpha$  is the asymptotic efficiency of the maximum likelihood estimator.

Finally observe, that the formulae (5) and (6) clearly imply that in the latter problem (as well as in any problem having  $Z_\infty$  as limiting likelihood ratio) the best choice of the maximum likelihood estimator is  $\alpha = 1/2$ , and that the so chosen maximum likelihood estimator is asymptotically efficient. This choice was also suggested by Chan and Kutoyants in [2] for problems having  $Z_\gamma$  as limiting likelihood ratio. For large values of  $\gamma$  this suggestion is confirmed by our asymptotic results. However, we see that for small values of  $\gamma$  the choice of  $\alpha$  will not be so important, since the limits in Theorem 1 do not depend on  $\alpha$ .

### 3 Proofs of the lemmas

In this section we give some hints on how to prove the lemmas. The details of the proofs can be found in [4].

First we consider Lemma 1. Note that the restrictions of the process  $\ln X_\gamma$  (as well as those of the process  $\ln Z_0$ ) on  $\mathbb{R}_+$  and on  $\mathbb{R}_-$  are mutually independent processes with stationary and independent increments. So, to obtain the convergence of all the finite-dimensional distributions, it is sufficient to show the weak convergence of one-dimensional distributions only, that is,

$$\ln X_\gamma(y) \Rightarrow \ln Z_0(y) = W(y) - \frac{|y|}{2}$$

for all  $y \in \mathbb{R}$ . Moreover, these processes being symmetric, it is sufficient to consider  $y \in \mathbb{R}_+$  only. Now, observing that conditionally to the  $\sigma$ -algebra related to the Poisson process  $\Pi_+$  the random variable  $\ln X_\gamma(y) = \gamma \sum_{k=1}^{\Pi_+(y/\gamma^2)} \epsilon_k^+ - \frac{\gamma^2}{2} \Pi_+(y/\gamma^2)$  is Gaussian with mean  $-\frac{\gamma^2}{2} \Pi_+(y/\gamma^2)$  and variance  $\gamma^2 \Pi_+(y/\gamma^2)$ , the characteristic function of  $\ln X_\gamma(y)$  can be calculated as  $\varphi_\gamma(t) = \mathbf{E} e^{-\frac{\gamma^2}{2}(it+t^2)\Pi_+(y/\gamma^2)}$ . Finally, noting that  $\Pi_+(y/\gamma^2)$  is a Poisson random variable of parameter  $y/\gamma^2$ , we get  $\ln \varphi_\gamma(t) = \frac{y}{\gamma^2} \left( e^{-\frac{\gamma^2}{2}(it+t^2)} - 1 \right) \rightarrow -\frac{y}{2}(it+t^2) = \ln \mathbf{E} e^{it(W(y) - \frac{|y|}{2})}$  as  $\gamma \rightarrow 0$ , and so Lemma 1 is proved.

The same techniques allow us to calculate  $\mathbf{E}X_\gamma^{1/2}(y) = \exp\left(\frac{|y|}{\gamma^2} \left(e^{-\frac{\gamma^2}{8}} - 1\right)\right)$ . This equality easily yields Lemma 3. Moreover, considering the cases of different signs of  $y_1$  and  $y_2$ , as well as taking into account the stationarity and the independence of increments of  $\ln X_\gamma$  on  $\mathbb{R}_+$  and on  $\mathbb{R}_-$ , Lemma 2 can also be deduced.

Regarding Lemma 4 let us consider, for example, the case  $y_1, y_2 \in \mathbb{R}_+$ ,  $y_1 \geq y_2$ . Denoting  $\Delta = y_1 - y_2$  and using the same type argument as above, we get

$$\mathbf{P}\left\{|\ln X_\gamma(y_1) - \ln X_\gamma(y_2)| > \varepsilon\right\} \leq \frac{1}{\varepsilon^2} \mathbf{E}|\ln X_\gamma(\Delta)|^2 = \frac{1}{\varepsilon^2} (\beta(\gamma) \Delta + \Delta^2/4)$$

where  $\beta(\gamma) = 1 + \gamma^2/4 \rightarrow 1$  as  $\gamma \rightarrow 0$ , and hence the result follows.

It remains to prove Lemma 5. First, taking into account the symmetry of the process  $\ln X_\gamma$  and using the stationarity and the independence of its increments on  $\mathbb{R}_+$ , we get

$$\mathbf{P}\left\{\sup_{|y|>A} X_\gamma(y) > e^{-bA}\right\} \leq 2e^{bA/2} \mathbf{E}X_\gamma^{1/2}(A) \mathbf{E}\sup_{z>0} X_\gamma^{1/2}(z).$$

Further, observing that the random process  $S_n = \sum_{k=1}^n \varepsilon_k^+$ ,  $n \in \mathbb{N}$ , has the same law as the restriction on  $\mathbb{N}$  of a standard Brownian motion  $W$ , we obtain

$$\mathbf{E}\sup_{z>0} X_\gamma^{1/2}(z) = \mathbf{E}\exp\left(\frac{1}{2} \sup_{n>0} \left(\gamma S_n - \frac{n\gamma^2}{2}\right)\right) \leq \mathbf{E}\exp\left(\frac{1}{2} \sup_{t>0} (W(t) - t/2)\right) = 2,$$

where the last equality is due to the fact that  $\sup_{t>0} (W(t) - t/2)$  is an exponential random variable of parameter 1 (see, for example, [1]). Now the proof can be concluded using Lemma 3.

## 4 Final remarks

In conclusion let us mention that in [2], the Gaussianity of the jumps of the limiting likelihood ratio process  $Z_\gamma$  is due to a Gaussianity assumption on the underlying threshold autoregressive model. Without such type assumption, many change-point type statistical models — for example, the multi-phase regression model considered by Fujii in [6] (and probably the threshold autoregressive model itself) — give raise to a more general compound Poisson type limiting likelihood ratio process  $Z_{\gamma,f}$ . This process is still an exponent of a two-sided compound Poisson process, but the jumps of the latter are no longer necessarily Gaussian. More precisely, it is given by

$$\ln Z_{\gamma,f}(x) = \begin{cases} \sum_{k=1}^{\Pi_+(x)} \ln \frac{f(\varepsilon_k^+ + \gamma)}{f(\varepsilon_k^+)}, & \text{if } x \geq 0, \\ \sum_{k=1}^{\Pi_-(-x)} \ln \frac{f(\varepsilon_k^- - \gamma)}{f(\varepsilon_k^-)}, & \text{if } x \leq 0, \end{cases}$$

where  $\gamma > 0$ ,  $\Pi_+$  and  $\Pi_-$  are two independent Poisson processes of intensity 1 on  $\mathbb{R}_+$ , and  $\varepsilon_k^\pm$  are independent random variables with density  $f$ , mean 0 and variance 1, which are also independent of  $\Pi_\pm$ . Our guess is that the results of the present paper hold in this general situation under some regularity conditions on  $f$ . The proofs of Lemmas 1–4 are easily adapted, and the one of Lemma 5 is actually a work in progress.

## References

1. Borodin, A.N. and Salminen, P.: Handbook of Brownian motion — facts and formulae. Probability and its Applications, Birkhäuser Verlag, Basel (2002)
2. Chan, N.H. and Kutoyants, Yu.A.: On parameter estimation of threshold autoregressive models (2009) Available via ArXiv. <http://arxiv.org/abs/1003.3800>
3. Dachian, S.: On limiting likelihood ratio processes of some change-point type statistical models. *J. Stat. Plan. Infer.* (2010) doi:10.1016/j.jspi.2010.03.030
4. Dachian, S. and Negri I.: On compound Poisson type limiting likelihood ratio process arising in some change-point models. WP n.11/MS, Dep. IIMM (2009) Available on line. <http://wwwdata.unibg.it/dati/bacheca/250/39592.pdf>
5. Deshayes, J. and Picard, D.: Lois asymptotiques des tests et estimateurs de rupture dans un modèle statistique classique. *Ann. Inst. H. Poincaré Probab. Statist.* **20**, no. 4, pp. 309–327 (1984)
6. Fujii, T.: On weak convergence of the likelihood ratio process in multi-phase regression models. *Statist. Probab. Lett.* **78**, no. 14, pp. 2066–2074 (2008)
7. Gihman, I.I. and Skorohod, A.V.: The theory of stochastic processes I. Springer-Verlag, New York (1974)
8. Golubev, G.K.: Computation of the efficiency of the maximum-likelihood estimator when observing a discontinuous signal in white noise. *Problems Inform. Transmission.* **15**, no. 3, pp. 61–69 (1979)
9. Höpfner, R. and Kutoyants, Yu.A.: Estimating discontinuous periodic signals in a time inhomogeneous diffusion (2009) Available via ArXiv. <http://arxiv.org/abs/0903.5061>
10. Ibragimov, I.A. and Khasminskii, R.Z.: Estimation of a parameter of a discontinuous signal in a white Gaussian noise. *Problems Inform. Transmission.* **11**, no. 3, pp. 31–43, (1975)
11. Ibragimov, I.A. and Khasminskii, R.Z.: Statistical estimation. Asymptotic theory. Springer-Verlag, New York (1981)
12. Küchler, U. and Kutoyants, Yu.A.: Delay estimation for some stationary diffusion-type processes. *Scand. J. Statist.* **27**, no. 3, pp. 405–414 (2000)
13. Kutoyants, Yu.A.: Parameter estimation for stochastic processes. Armenian Academy of Sciences, Yerevan, in Russian (1980). Translation of revised version, Heldermann-Verlag, Berlin (1984)
14. Kutoyants, Yu.A.: Identification of dynamical systems with small noise. Mathematics and its Applications **300**, Kluwer Academic Publishers Group, Dordrecht (1994)
15. Kutoyants, Yu.A.: Statistical inference for ergodic diffusion processes. Springer Series in Statistics, Springer-Verlag, London (2004)
16. Rubin, H. and Song, K.-S.: Exact computation of the asymptotic efficiency of maximum likelihood estimators of a discontinuous signal in a Gaussian white noise. *Ann. Statist.* **23**, no. 3, pp. 732–739 (1995)
17. Terent'yev, A.S.: Probability distribution of a time location of an absolute maximum at the output of a synchronized filter. *Radioengineering and Electronics* **13**, no. 4, pp. 652–657 (1968)