

More on a new three-parameter distribution for non-negative variables

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Abstract In this paper we analyze some features of M.M. Zenga's new three parameter density function. We report the expressions of the moments, the distribution function, the mean deviation and of Zenga's point inequality measure at the mean. Moreover we report some graphs of the density function, of Zenga's inequality curve and of the Lorenz curve. The new density takes on a wide variety of shapes, and preliminary applications suggest that it provides a good fit to income and other economic size distributions.

Key words: Statistical modeling, non-negative variables, positive asymmetry, parietan right tail, beta function.

1 Introduction

Recently M. M. Zenga (2009) has proposed a new three parameter density function $f(x : \mu; \alpha; \theta)$, ($\mu > 0; \alpha > 0; \theta > 0$), for non negative variables. The density has been obtained as a mixture of Poliscchio's (2008) following truncated Pareto density

$$f(x : \mu; k) = \begin{cases} \frac{\sqrt{\mu}}{2} k^{0.5} (1-k)^{-1} x^{-1.5}, & \mu k \leq x \leq \frac{\mu}{k}; \\ 0, & \text{otherwise,} \end{cases} \quad \mu > 0, 0 < k < 1; \quad (1)$$

with weights given by the beta density

$$g(k : \alpha; \theta) = \begin{cases} \frac{k^{\alpha-1} (1-k)^{\theta-1}}{B(\alpha; \theta)}, & 0 < k < 1; \\ 0, & \text{otherwise,} \end{cases} \quad \theta > 0, \alpha > 0; \quad (2)$$

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where $B(\alpha; \theta)$ is the beta function.

2 Some characteristics of the new density

M.M. Zenga (2009) has shown that for $\theta > 0$ the density $f(x : \mu; \alpha; \theta)$ is given by

$$f(x : \mu; \alpha; \theta) = \begin{cases} \frac{1}{2\mu B(\alpha; \theta)} \left(\frac{x}{\mu}\right)^{-1.5} \int_0^{\frac{x}{\mu}} k^{\alpha+0.5-1} (1-k)^{\theta-2} dk, & 0 < x < \mu \\ \frac{1}{2\mu B(\alpha; \theta)} \left(\frac{\mu}{x}\right)^{1.5} \int_0^{\frac{\mu}{x}} k^{\alpha+0.5-1} (1-k)^{\theta-2} dk, & \mu < x. \end{cases}$$

Note that:

$$\lim_{x \rightarrow \mu} f(x : \mu; \alpha; \theta) = \begin{cases} \frac{B(\alpha+0.5; \theta-1)}{2\mu B(\alpha; \theta)}, & \text{if } \theta > 1 \\ \infty, & \text{if } 0 < \theta \leq 1; \end{cases}$$

$$\lim_{x \rightarrow 0} f(x : \mu; \alpha; \theta) = \begin{cases} 0, & \text{for } \alpha > 1 \\ \frac{1}{3} \frac{\theta}{\mu}, & \text{for } \alpha = 1 \\ \infty, & \text{for } 0 < \alpha < 1. \end{cases}$$

Figures 1 and 2 report graphs of the new density function for some parameter values.

The moment of order r is finite if and only if $\alpha + 1 > r$ and is given by

$$E(X^r) = \frac{\mu^r}{(2r-1)B(\alpha; \theta)} \sum_{i=1}^{2r-1} B(\alpha-r+i; \theta).$$

This yields

$$E(X) = \mu$$

and

$$Var(X) = \frac{\mu^2}{3} \frac{\theta(\theta+1)}{(\alpha-1)(\alpha+\theta)}.$$

The distribution function $F(x : \mu; \alpha; \theta)$ is given by

$$\frac{1}{B(\alpha; \theta)} \left\{ \sum_{i=1}^{\infty} IB\left(\frac{x}{\mu} : \alpha+i-1; \theta\right) - \left(\frac{x}{\mu}\right)^{-0.5} \sum_{i=1}^{\infty} IB\left(\frac{x}{\mu} : \alpha+i-\frac{1}{2}; \theta\right) \right\} \quad (1)$$

for $0 < x \leq \mu$, and by

$$1 - \frac{1}{B(\alpha; \theta)} \left\{ \left(\frac{\mu}{x} \right)^{0.5} \sum_{i=1}^{\infty} IB\left(\frac{\mu}{x} : \alpha + i - 0.5; \theta\right) - \sum_{i=1}^{\infty} IB\left(\frac{\mu}{x} : \alpha + i; \theta\right) \right\} \quad (2)$$

for $x > \mu$. In formulae (1) and (2) we used the symbol $IB(x; \alpha; \theta)$ to indicate the incomplete beta integral, i.e.

$$IB(x : \alpha; \theta) = \int_0^x z^{\alpha-1} (1-z)^{\theta-1} dz.$$

Explicit formulae for the mean deviation and for Zenga's point inequality (M.M. Zenga, 2007) are also known. We have

$$E(|X - \mu|) = 2\mu[2F(1 : 1; \alpha; \theta) - 1]$$

and

$$A(\mu) = 1 - \frac{E(X|X \leq \mu)}{E(X|X > \mu)} = \left\{ \frac{1 - F(1 : 1; \alpha; \theta)}{F(1 : 1; \alpha; \theta)} \right\}^2.$$

Finally, we report in figure 3 and 4 the graphs of Zenga's (2007) $I(p)$ curve and of the Lorenz curve.

3 Conclusions

M.M. Zenga (2009) proposed a new three parameter density function $f(x : \mu; \alpha; \theta)$, ($\mu > 0; \alpha > 0; \theta > 0$) for non negative random variables X . For $\theta > 1$ M.M. Zenga (2009) has obtained the expressions of: the distribution function, the moments, the mean deviation and M.M. Zenga's (2009) point inequality $A(x)$ at $x = \mu$. Many graphs of $f(x : \mu; \alpha; \theta)$ for $\theta \geq 2$ are reported in M.M. Zenga (2009), too. According to this paper we observe that for this new distribution:

- a) the parameter μ is equal to the expectation;
- b) the right tail is Paretian and the asymmetry is positive;
- c) the shapes of the density function $f(x : \mu; \alpha; \theta)$ are broader than those of the more traditional models used for the distribution of income by size such as Dagum's distribution.

In this paper by using a new approach, we obtain for the general case $\theta > 0$ different expressions of: the density, the truncated and ordinary moments, the mean deviation and M.M. Zenga's point inequality. It can be shown that the new expressions reported in this paper for $\theta > 0$ are equivalent to those obtained previously for $\theta > 1$ by M.M. Zenga (2009). The graphs of the density function for $0.5 \leq \theta \leq 1.5$ confirm that the new density has very "flexible" shapes. The graphs reported in this paper of: the Lorenz curve $L(p)$ and the Zenga curve $I(p)$ suggest that the new density can be utilized to represent income, wealth, financial and actuarial as well as failure time distributions. Some first exploratory applications of the new density on real income

distributions encourage to investigate on the estimation of the parameters of the new density.

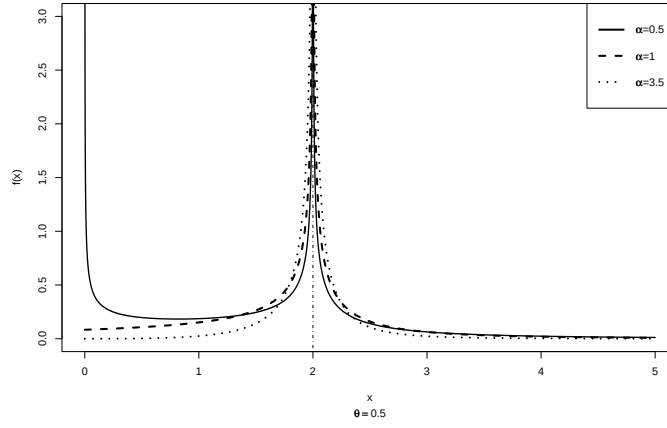


Fig. 1 Graphs of $f(x : 2; \alpha; 0.5)$ for $\theta = 0.5$ and $\alpha = 0.5; 1; 3.5$

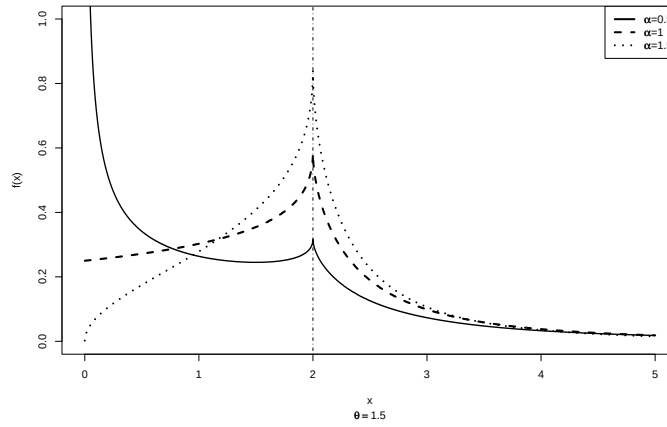


Fig. 2 Graphs of $f(x : 2; \alpha; 1.5)$ for $\theta = 1.5$ and $\alpha = 0.5; 1; 1.5$

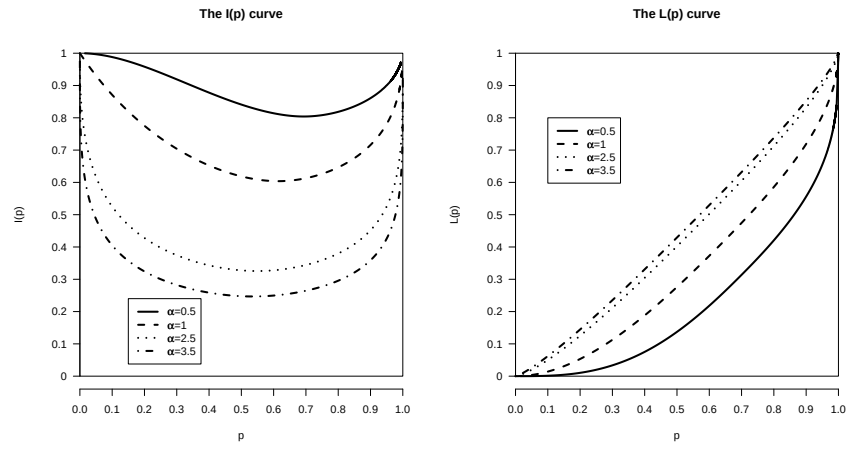


Fig. 3 Graphs of $I(p)$ and $L(p)$ for $\theta = 1$ and $\alpha = 0.5; 1; 2.5; 3.5$

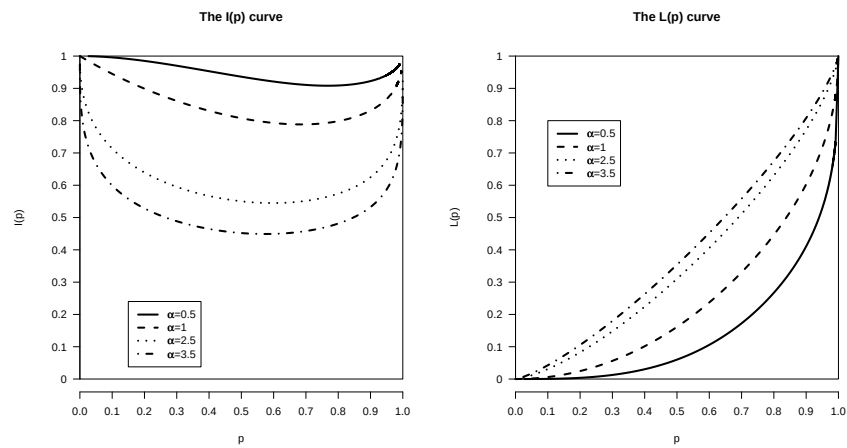


Fig. 4 Graphs of $I(p)$ and $L(p)$ for $\theta = 2.5$ and $\alpha = 0.5; 1; 2.5; 3.5$

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