

Comparing Unrelated Question Models from a Privacy Protection Perspective

Sabrina Giordano, Giovanni Latorre and Pier Francesco Perri

Abstract In this paper the problem of estimating the proportion π_A of people bearing a sensitive, stigmatizing attribute A is considered. Three dichotomous unrelated question models are discussed and their performance is evaluated by taking into account both efficiency and respondent's privacy protection. Analytical comparisons are carried out by comparing the variances of estimators of π_A at the same level of the jeopardy measures introduced by Leysieffer and Warner (1976). The conditions for the equivalence/superiority of a randomization design are given in closed form.

Key words: Sample Design and Estimation, Privacy Protection, Simmons' Model, Efficiency, Jeopardy Measures.

1 Introduction

Surveys on controversial or highly personal and stigmatizing questions concerning, for example, drug addiction, abortion, rape, xenophobia and fraud can lead to unreliable data when people interviewed are asked to answer directly. Consider, for instance, the case in which people are asked to give a *yes* or *no* response to the question “*Have you ever used drugs?*”. Such direct questioning can be met with refusal to cooperate or untruthful answers.

To induce greater cooperation and collect data on sensitive topics while ensuring anonymity to respondents, Warner (1965) introduced an ingenious tool known as *randomized response* (RR) technique. The population under study is divided into two mutually exclusive groups, one with and the other without a stigmatizing at-

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tribute, say A . In order to estimate the proportion π_A of population possessing A , a sample of n respondents is selected and each one is provided with a deck of cards. On each card is written one of two statements: “*I possess A*” or “*I do not possess A*”, in a proportion which is fixed in advance by the researcher. Respondents are asked to select a card and report a *yes* or *no* response depending on whether their true status - with respect to A - matches the statement on the selected card or not. Respondents are assumed to answer truthfully and, since they are instructed to not reveal to anyone the question selected, and both the interviewers and the researcher are unaware of the randomization device result, their true status remains unknown and thus privacy is protected.

Different variants of Warner’s model have been suggested so far. A first alternative, aimed at enhancing respondent’s cooperation, is known as *unrelated question model* or *Simmons’ model* (Horvitz et al. (1967); Greenberg et al. (1969)). The technique provides respondents with the opportunity of replying to one of two questions: “*I possess A*” or “*I possess B*”, B being an innocuous characteristic unrelated to A and possessed by a known proportion π_B of the population. Moreover, for the purpose of this paper, we mention Mangat et al. (1995) who designed an RR device by adding to the two types of cards in Warner’s design a number of blank cards. If a blank card is selected, respondents are asked to say *no*, irrespective of their true status. Bhargava and Singh (2000) considered the case when respondents are instructed to reply *yes* when a blank card occurs. Similar procedures have been adopted by Singh et al. (2003) and Perri (2008) for Simmons’ model.

Comparisons of different RR designs are usually only performed in terms of efficiency of the estimates, neglecting the degree of complexity and, above all, the level of confidentiality guaranteed. As pointed out by many authors (see, e.g. Guerriero and Sandri (2006)), a fair comparison should consider both efficiency and privacy: variances of the estimates are to be compared under equal levels of confidentiality. With this spirit, Lin (2005) compared Warner (1965), Mangat et al. (1995) and Bhargava and Singh (2000) models according to the efficiency/protection criterion. Starting from Lin (2005), in this paper we intend to compare the performance of the unrelated question models of Singh et al. (2003) and Perri (2008) with Simmons’ design. In so doing, we will consider the jeopardy measures proposed by Leysieffer and Warner (1976) and show the interrelation between respondent’s jeopardy and efficiency.

The paper is organized as follows. Section 2 introduces, in a general framework, the problem of estimating the proportion of people bearing a sensitive attribute when a dichotomous-response model is assumed. The unrelated question models earlier mentioned are discussed and their variance given. Section 3 is devoted to the protection of privacy: jeopardy measures are introduced and the relation between the variance of the estimators and the degree of privacy is pointed out. In Section 4, the unrelated question models are compared in terms of variances under equal levels of confidentiality and the optimal RR designs are provided. Section 5 concludes the work with some final comments.

2 Unrelated Question Models

Consider a dichotomous population consisting of a sensitive group A and its complement $\bar{A}$ in the unknown proportions π_A and $1 - \pi_A$. Consider a generic dichotomous RR design with a typical response R which can be *yes* (say, Y) or *no* (say, N). Let $P(R|A)$ and $P(R|\bar{A})$ be the conditional probabilities that a response R comes from a member of groups A and $\bar{A}$, respectively. These probabilities denote the *design probabilities* of an RR survey. They are controlled by the researcher and are known both to the interviewee and the interviewer. Without loss of generality, the probability of a *yes* response can be expressed as

$$\lambda = P(Y|A)\pi_A + P(Y|\bar{A})(1 - \pi_A) = [P(Y|A) - P(Y|\bar{A})]\pi_A + P(Y|\bar{A}). \quad (1)$$

To estimate π_A , a simple random sample of n respondents is drawn with replacement from the population. Sampled individuals are instructed to execute the RR device and give a *yes* or *no* response according to the outcome of the procedure. Let n_1 be the number of *yes* responses in the sample, $n_1 \sim \text{Bin}(n, \lambda)$, and $\hat{\lambda} = n_1/n$. Then, from (1), the unbiased *moment estimator* of π_A is

$$\hat{\pi}_A = \frac{\hat{\lambda} - P(Y|\bar{A})}{P(Y|A) - P(Y|\bar{A})} \quad (2)$$

which is defined if $P(Y|A) - P(Y|\bar{A}) \neq 0$ and has variance

$$V(\hat{\pi}_A) = \frac{\lambda(1 - \lambda)}{n[P(Y|A) - P(Y|\bar{A})]^2}. \quad (3)$$

The general expressions given above can be used for any RR design satisfying the working assumptions. We need only to specify the design probabilities. For instance, consider Simmons' model and let p and $1 - p$ be the *design parameters*, i.e. the probabilities of answering questions “*I possess A*” and “*I possess B*”, respectively. Then, $P(Y|A) = p + (1 - p)\pi_B$, $P(Y|\bar{A}) = (1 - p)\pi_B$. From (2) and (3), we get the expression for the estimator of π_A and its variance as $\hat{\pi}_S = [\hat{\lambda} - (1 - p)\pi_B]/p$ and

$$V(\hat{\pi}_S) = \frac{\pi_A(1 - \pi_A)}{n} + \frac{\pi_A(1 - p)(1 - 2\pi_B)}{np} + \frac{\pi_B(1 - p)[1 - (1 - p)\pi_B]}{np^2}. \quad (4)$$

In Singh et al.'s (2003) RR model (hereafter Design 1), cards show the statement “*I possess A*”, others “*I possess B*”, and the rest of the cards are left blank. These three types of cards occur in the proportion of q_1 , q_2 and q_3 (design parameters), with $\sum_{i=1}^3 q_i = 1$. The device works as in Simmons' with the exception that, in the case of a blank card being chosen, respondents will report *no*, no matter what their true status is. We observe that the procedure operates in the same manner as the strategy proposed by Mangat et al. (1995) for modifying Warner's method. The design probabilities are given by $P(Y|A) = q_1 + q_2\pi_B$ and $P(Y|\bar{A}) = q_2\pi_B$. According to the randomization procedure, π_A can be estimated by $\hat{\pi}_1 = (\hat{\lambda} - q_2\pi_B)/q_1$ with

$$V(\hat{\pi}_1) = \frac{\pi_A(1 - \pi_A)}{n} + \frac{\pi_A(1 - q_1 - 2q_2\pi_B)}{nq_1} + \frac{q_2\pi_B(1 - q_2\pi_B)}{nq_1^2}. \quad (5)$$

A variant of Design 1 by Perri (2008) (hereafter Design 2) provides a *yes* response when a blank card is selected. Let p_1 and p_2 be the probabilities of selecting a card with statements A and B , and p_3 the probability of a blank card, $\sum_{i=1}^3 p_i = 1$. Then, $P(Y|A) = p_1 + p_2\pi_B + p_3$, $P(Y|\bar{A}) = p_2\pi_B + p_3$, $\hat{\pi}_2 = (\hat{\lambda} - p_2\pi_Y - p_3)/p_1$ and

$$V(\hat{\pi}_2) = \frac{\pi_A(1 - \pi_A)}{n} + \frac{\pi_A[p_2(1 - 2\pi_B) - p_3]}{np_1} + \frac{(p_3 + p_2\pi_B)(1 - p_3 - p_2\pi_B)}{np_1^2}. \quad (6)$$

The performance of these unrelated question models has been evaluated in Perri (2008) by comparing $V(\hat{\pi}_S)$, $V(\hat{\pi}_1)$ and $V(\hat{\pi}_2)$ under the assumption $p = q_1 = p_1$ and $q_3 = p_3$. The comparison has shown that Design 2 outperforms Simmons' model as well as Design 1 in terms of efficiency under certain conditions only. Nonetheless, it is generally observed that the efficiency of a strategy increases as the level of confidentiality decreases. Therefore, we shall now proceed to compare the considered designs taking both efficiency and privacy protection into account.

3 Measuring Respondent's Privacy Protection

One way to evaluate different RR models is by comparing the variances when the degree of privacy protection is held constant. Many criteria for measuring the degree of confidentiality induced by an RR design have been proposed in the literature. Here, we focus on the *jeopardy measures* by Leysieffer and Warner (1976)

$$g(Y, A) = \frac{P(Y|A)}{P(Y|\bar{A})}, \quad g(N, \bar{A}) = \frac{P(N|\bar{A})}{P(N|A)} = \frac{1 - P(Y|\bar{A})}{1 - P(Y|A)}. \quad (7)$$

It is important to note that g is a function of the design probabilities, but not of π_A . Values of $g(Y, A)$ [$g(N, \bar{A})$] greater than unity indicate that response Y [N] is jeopardizing with respect to A [$\bar{A}$], in the sense that with this response members of group A [$\bar{A}$] are more likely to belong to group A [$\bar{A}$] rather than to group $\bar{A}$ [A]. Therefore, for respondents genuinely belonging to group A [$\bar{A}$], the RR device yields an increased posterior probability $P(A|Y)$ [$P(\bar{A}|N)$] that they really belong to group A [$\bar{A}$], and this may induce their suspicions.

The jeopardy measures are linked to the efficiency of the estimates. In fact, for designs where $P(Y|A) - P(Y|\bar{A}) \neq 0$ holds, it can be shown (Leysieffer and Warner (1976)) that

$$V(\hat{\pi}) = \frac{\pi_A(1 - \pi_A)}{n} + \frac{\pi_A g(Y, A) + (1 - \pi_A)g(N, \bar{A})}{n[g(Y, A) - 1][g(N, \bar{A}) - 1]}. \quad (8)$$

For $g(Y, A) > 1$ and $g(N, \bar{A}) > 1$, $V(\hat{\pi})$ is a decreasing function of $g(Y, A)$, $g(N, \bar{A})$ or both. The general expressions for g given in (7) can be easily worked out for any RR design by substituting the proper design probabilities. For Simmons' model the jeopardy measures take the form

$$g_S(Y, A) = \frac{p + (1-p)\pi_B}{(1-p)\pi_B}, \quad g_S(N, \bar{A}) = \frac{1 - (1-p)\pi_B}{(1-p)(1-\pi_B)} \quad (9)$$

whereas, for Designs 1 and 2 we have

$$g_1(Y, A) = \frac{q_1 + q_2\pi_B}{q_2\pi_B}, \quad g_1(N, \bar{A}) = \frac{1 - q_2\pi_B}{1 - q_1 - q_2\pi_B} \quad (10)$$

$$g_2(Y, A) = \frac{1 - p_2(1 - \pi_B)}{p_2\pi_B + p_3}, \quad g_2(N, \bar{A}) = \frac{1 - p_2\pi_B - p_3}{p_2(1 - \pi_B)}. \quad (11)$$

It is easy to verify that $g > 1$ for each considered randomization design, meaning that respondents' privacy is jeopardized both with a *yes* or a *no* answer.

4 Comparisons of the Unrelated Question Procedures

For proper design comparisons, both efficiency and privacy protection should be taken into account. Therefore, we compare the three unrelated question procedures with respect to the efficiency at the same level of jeopardy.

Let K_1 and K_2 be the maximal allowable values, both above unity, of the jeopardy measures $g(Y, A)$ and $g(N, \bar{A})$, respectively. Allowing jeopardy degrees less than K_1 and K_2 would assure respondents of a higher degree of protection, but this would necessarily increase the variance of the estimators as shown in (8).

The optimal choice of the design parameters for different techniques is drawn so that $g(Y, A) = K_1$ and $g(N, \bar{A}) = K_2$. The variances implied by these choices are then compared and the design with the smallest value is judged the most efficient under the given privacy restrictions.

We proceed now to compare the unrelated question procedures under various choices for K_1 and K_2 . First, suppose that K_1 and K_2 are equal and finite. The same degree $K_1 = K_2 = K$ of the two jeopardy measures (9) for Simmons' model is allowed if $\pi_B = 0.5$ and $p = (K - 1)/(K + 1)$. But, under this choice of π_B , it is clear from (10) that in Design 1 the privacy constraints $K_1 = K_2 = K$ would be met only if $q_1 = 0$ or $q_3 = 0$ and, for Design 2, only if $p_1 = 0$ and $p_3 = 0$ by (11). Such design parameters may be of little relevance as they provide mechanisms which instruct respondents never to answer question A ($q_1 = p_1 = 0$), or reduce Designs 1 and 2 to the Simmons' model ($q_3 = p_3 = 0$).

Next, assume $K_1 \neq K_2$ both finite. Under this condition, Simmons' model cannot be improved either by Design 1 or 2. The following proposition shows that the three

methods provide the same efficiency when compared at the same level of privacy protection.

Proposition 1 *Simmons' model and Designs 1 and 2 are equally efficient under the condition of equal levels of the jeopardy measures, $g(Y, A) = K_1$ and $g(N, \bar{A}) = K_2$, $K_1 \neq K_2$ both finite.*

Proof. We start by comparing Simmons' model and Design 1. The optimal choice for π_B and p of Simmons' model follows by $g_S(Y, A) = K_1$ and $g_S(N, \bar{A}) = K_2$

$$\pi_B = \frac{K_2 - 1}{K_1 + K_2 - 2}, \quad p = \frac{(K_1 - 1)(K_2 - 1)}{K_1 K_2 - 1}. \quad (12)$$

With this choice, the variance expression (4) is reduced to

$$V(\hat{\pi}_S) = \frac{\pi_A(1 - \pi_A)}{n} + \frac{K_1 \pi_A + K_2(1 - \pi_A)}{n(K_2 - 1)(K_1 - 1)}.$$

Similarly, the constraints $g_1(Y, A) = K_1$ and $g_1(N, \bar{A}) = K_2$ lead to a Design 1 with parameters

$$q_1 = \frac{(K_1 - 1)(K_2 - 1)}{K_1 K_2 - 1}, \quad q_2 = \frac{K_2 - 1}{\pi_B(K_1 K_2 - 1)} \quad (13)$$

and consequently $q_3 = 1 - q_1 - q_2$. From (5), the resulting variance of $\hat{\pi}_1$ is $V(\hat{\pi}_1) = \frac{\pi_A(1 - \pi_A)}{n} + \frac{K_1 \pi_A + K_2(1 - \pi_A)}{n(K_2 - 1)(K_1 - 1)}$ which coincides with $V(\hat{\pi}_S)$. Thus, the two methods are equally efficient and protective. Now, we compare Simmons' model and Design 2. Setting $g_2(Y, A) = K_1$ and $g_2(N, \bar{A}) = K_2$ yields the best choice

$$p_2 = \frac{K_1 - 1}{(1 - \pi_B)(K_1 K_2 - 1)}, \quad p_3 = \frac{K_1 \pi_B + K_2(\pi_B - 1) - 2\pi_B + 1}{(\pi_B - 1)(K_1 K_2 - 1)}$$

and $p_1 = 1 - p_2 - p_3$. Then, by (6), $V(\hat{\pi}_2) = \frac{\pi_A(1 - \pi_A)}{n} + \frac{K_1 \pi_A + K_2(1 - \pi_A)}{n(K_2 - 1)(K_1 - 1)}$, that is equal to the Simmons' one. Hence the proof. $\square$

As regards another case, assume K_1 finite but K_2 unlimited. This refers to a problem in which there is no jeopardy in a *no* answer (Greenberg et al. (1969)). Under this condition, we find the optimal choice of the design parameters so that the three models turn out to be equally efficient with same jeopardy level. This result is formally stated in the proposition below.

Proposition 2 *Designs 1, 2 and Simmons' model offer equal efficiency when compared at the same level of protection of respondent's privacy, $g(Y, A) = K_1 < \infty$ and $g(N, \bar{A}) = K_2$ unlimited.*

Proof. Allowing $K_2 \rightarrow \infty$, the optimal choice of the design parameters for Simmons' model is $\pi_B = 1$ and $p = \frac{K_1 - 1}{K_1}$ by (12). Under these conditions, the variance of Simmons' estimator becomes $V(\hat{\pi}_S) = \frac{\pi_A(1 - \pi_A)}{n} + \frac{(1 - \pi_A)}{n(K_1 - 1)}$.

Taking the limits of (13) for $K_2 \rightarrow \infty$, provided that $\pi_B = 1$, Design 1 parameters assume the values $q_1 = \frac{K_1-1}{K_1}$, $q_2 = \frac{1}{K_1}$ and, thus, $q_3 = 0$. Hence, the resulting model is merely reduced to that of Simmons. Analogously, the optimal Design 2 parameters are $p_2 = 0$, $p_3 = \frac{1}{K_1}$ and, therefore, $p_1 = \frac{K_1-1}{K_1}$. With these parameters, $V(\hat{\pi}_2)$ coincides with that of the previous two unrelated question models, $\forall \pi_B \in (0, 1]$. Therefore, as regards efficiency it follows that all the three strategies are equivalent when forced to offer the same level of privacy in the case where a *no* response is non-jeopardizing. $\square$

Practical considerations on the protection of confidentiality can induce the researcher to consider the level of $g(N, \bar{A})$ not of major concern and give more importance to the control of $g(Y, A)$. In this context it is useful to regard $g(Y, A)$ as fixed and compare the variances and $g(N, \bar{A})$ values of RR methods sharing a common value of $g(Y, A)$. The following proposition focuses on this point.

Proposition 3 *Simmons' model is more efficient than Design 1 under the privacy restriction of equal jeopardy yes-measure, $g_S(Y, A) = g_1(Y, A)$.*

Proof. The condition $g_S(Y, A) = g_1(Y, A)$ ensures that $p = \frac{q_1}{q_1+q_2}$. By substituting this parameter in the expression of $g_S(N, \bar{A})$, it follows that $g_S(N, \bar{A}) > g_1(N, \bar{A})$, and consequently $V(\hat{\pi}_S) < V(\hat{\pi}_1)$ holds by (8). $\square$

Proposition 4 *Design 2 is more efficient than Simmons' model under the privacy restriction of equal jeopardy yes-measure, $g_S(Y, A) = g_2(Y, A)$.*

Proof. See proof of Proposition 3. $\square$

On the other hand, one can fix a maximal allowable degree of $g(N, \bar{A})$ and describe how the three unrelated question models behave in terms of efficiency. Comparisons of the procedures are illustrated in the next propositions.

Proposition 5 *Design 1 is more efficient than Simmons' model under the privacy restriction of equal jeopardy no-measure, $g_S(N, \bar{A}) = g_1(N, \bar{A})$.*

Proof. Equating $g_S(N, \bar{A})$ and $g_1(N, \bar{A})$ provides $p = \frac{q_1(\pi_B-1)}{\pi_B(q_1+q_2)-1}$. It is easy to prove that the measure $g_S(Y, A)$, expressed in terms of this optimal p , is always less than $g_1(Y, A)$ ensuring that $V(\hat{\pi}_S) > V(\hat{\pi}_1)$ by (8). Thus, Design 1 outperforms Simmons' strategy in terms of efficiency when the measured protection to *no*-answer in the two competitive models is equal. $\square$

Proposition 6 *Simmons' model is more efficient than Design 2 under the privacy restriction of equal jeopardy no-measure, $g_S(N, \bar{A}) = g_2(N, \bar{A})$.*

Proof. See proof of Proposition 5. $\square$

The discussed comparisons reveal that Simmons' model, compared with the two alternative unrelated question designs, plays an intermediate role in terms of efficiency and degree of one jeopardy measure when the same value of the other jeopardy function is assumed. More formally, the results are summarized in the corollary below.

Corollary 1 *Suppose $g_S(Y, A) = g_1(Y, A) = g_2(Y, A)$, it holds that*

$$g_2(N, \bar{A}) > g_S(N, \bar{A}) > g_1(N, \bar{A}) \quad \text{and} \quad V(\hat{\pi}_1) > V(\hat{\pi}_S) > V(\hat{\pi}_2).$$

Moreover, suppose $g_S(N, \bar{A}) = g_1(N, \bar{A}) = g_2(N, \bar{A})$, it turns out that

$$g_1(Y, A) > g_S(Y, A) > g_2(Y, A) \quad \text{and} \quad V(\hat{\pi}_2) > V(\hat{\pi}_S) > V(\hat{\pi}_1).$$

5 Concluding Remarks

The degree of privacy protection is an important aspect of the RR devices. Greater privacy implies, in general, an increase in terms of variance of the estimate. Therefore, one way of comparing different RR designs is to compare variances when a required level of privacy is ensured to respondents. With this in mind, we have evaluated the performance of three unrelated question models by paralleling the work by Lin (2005) where Warner's (1965) model and two of its variants have been compared in terms of efficiency/privacy. Questions remain open: (i) the analysis of the considered models on the basis of privacy measures out the jeopardy measures by Leysieffer and Warner (1976), and (ii) how much importance should be given to privacy protection. In fact, as it may be questionable whether the respondent is able to measure the extent of privacy disclosure offered by an RR design, some researchers would prefer to use designs privileging more efficient estimates rather than those guaranteeing high privacy protection.

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