

Asymptotic properties of numbers of observations near order statistics

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Abstract In this paper we discuss the joint limiting behavior of numbers of observations near order statistics. We show that the numbers of observations in two-sided neighborhoods of some upper and lower order statistics are asymptotically independent as the sample size tends to infinity. We also establish the multivariate limiting distribution of random variables describing numbers of observations near two or more central order statistics. Then we generalize the above mentioned results to the case of randomly indexed samples.

Key words: probability and stochastic processes, near order statistic observations, randomly indexed samples; weak limit theorems; asymptotic independence

1 Introduction

Let $(X_n, n \geq 1)$ be a sequence of independent and identically distributed (iid) random variables (rv's) with continuous cumulative distribution function (cdf) F , and $X_{1:n} \leq \dots \leq X_{n:n}$ be the order statistics based on the sample $(X_1, \dots, X_n)$.

For $k \in \{1, 2, \dots, n\}$ and $a > 0$, by $K_-(n, k, a)$ and $K_+(n, k, a)$ let us denote the numbers of observations that fall into the open left and right neighborhoods of the k -th order statistic, respectively, that is

$$K_-(n, k, a) = \#\{j \in \{1, \dots, n\} : X_j \in (X_{k:n} - a, X_{k:n})\}$$

and

$$K_+(n, k, a) = \#\{j \in \{1, \dots, n\} : X_j \in (X_{k:n}, X_{k:n} + a)\}.$$

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Exact and asymptotic properties of the rv's $K_-(n, k, a)$ and $K_+(n, k, a)$ as $n \rightarrow \infty$ and k or $n - k$ is held fixed have been investigated by many authors. Pakes and Steutel (1997) and Li (1999) studied the behavior of the number of observations near the sample maximum, $K_-(n, n, a)$. Generalizations of this result to the number of near k -extremes, $K_-(n, n - k, a)$, can be found in Pakes and Li (1998), Pakes (2000), Hu and Su (2003), and Balakrishnan and Stepanov (2005). The latter authors also considered the joint behavior of the rv's $K_-(n, n - k, a)$ and $K_+(n, n - k, b)$ and established that they are asymptotically independent under suitable conditions. Dembińska and Balakrishnan (2008) discussed the asymptotic independence of pairs of the rv's $K_-(n, k, a)$, $K_+(n, k, b)$, $K_-(n, n - r, c)$ and $K_+(n, n - r, d)$.

Dembińska, Stepanov and Wesołowski (2007) were the first to study the numbers of observations near a sample quantile, $K_-(n, k_n, a)$ and $K_+(n, k_n, b)$, where $k_n/n \rightarrow \alpha \in (0, 1)$. They showed that if F is strictly increasing in a neighborhood of the α -th quantile, then for any fixed $a > 0$,

$$K_-(n, k_n, a) \xrightarrow{P} \infty \quad \text{and} \quad K_+(n, k_n, a) \xrightarrow{P} \infty \quad \text{as} \quad n \rightarrow \infty, \quad (1)$$

where $\xrightarrow{P}$ denotes convergence in probability. In order to obtain nondegenerate limit laws, they allowed a to vary with n and found sequences $(a_n)_{n \geq 1}$ and $(b_n)_{n \geq 1}$ such that the rv's $K_-(n, k_n, a_n)$ and $K_+(n, k_n, b_n)$ both converge in distribution to Poisson under some suitable regularity conditions.

The aim of this paper is to generalize the results given by Dembińska and Balakrishnan (2008) and Dembińska, Stepanov and Wesołowski (2007) to the multidimensional case and then to give their extensions to the case of randomly indexed samples. In Section 2, we establish the limiting distribution of the random vector

$$(K_-(n, k, a), K_+(n, k, b), K_-(n, n - r, c), K_+(n, n - r, d)) \quad (2)$$

provided that $l_F := \inf\{x \in \mathbb{R} : F(x) > 0\} = -\infty$, $r_F := \sup\{x \in \mathbb{R} : F(x) < 1\} = \infty$ and $\bar{\beta}(a)$, $\bar{\beta}(b)$, $\beta(c)$, $\beta(d)$ all exist and belong to $(0, 1)$, where

$$\begin{aligned} \beta(y) &:= \lim_{x \rightarrow \infty} \frac{1 - F(x)}{1 - F(x - y)} = \lim_{x \rightarrow \infty} \frac{1 - F(x + y)}{1 - F(x)}, \\ \bar{\beta}(y) &:= \lim_{x \rightarrow -\infty} \frac{F(x)}{F(x + y)} = \lim_{x \rightarrow -\infty} \frac{F(x - y)}{F(x)}. \end{aligned}$$

We also determine the limiting law of random vectors $\mathbb{L}(n, k, \mathbf{a}) = (L_1, \dots, L_s)^T$ and $\mathbb{M}(n, k, \mathbf{a}) = (M_1, \dots, M_s)^T$ as $n \rightarrow \infty$, where $\mathbf{a} = (a_0, \dots, a_s)$,

$$\begin{aligned} L_j &= \#\{j \in \{1, \dots, n\} : X_j \in (X_{k:n} - a_j, X_{k:n} - a_{j-1})\}, \\ M_j &= \#\{j \in \{1, \dots, n\} : X_j \in (X_{k:n} + a_{j-1}, X_{k:n} + a_j)\}, \quad j = 1, \dots, s \end{aligned} \quad (3)$$

and $0 = a_0 < a_1 < \dots < a_s$ are real numbers. Next in Section 3, we consider the asymptotic joint behavior of the numbers of observations near central order statistics. We prove that if $\lim_{n \rightarrow \infty} k_n^{(i)}/n = \alpha_i \in (0, 1)$, $i = 1, \dots, s$, and all α_i 's are distinct,

then for special choices of sequences $\{a_n^{(i)}, n \geq 1\}$ and $\{b_n^{(i)}, n \geq 1\}$, $i = 1, \dots, s$, the rv's $K_-(n, k_n^{(1)}, a_n^{(1)})$, $K_+(n, k_n^{(1)}, b_n^{(1)})$, $\dots$, $K_-(n, k_n^{(s)}, a_n^{(s)})$, $K_+(n, k_n^{(s)}, b_n^{(s)})$ are asymptotically independent. Finally in Section 4, we give extensions of the results of previous sections to the case of randomly indexed samples.

2 Observations near extreme order statistics

This section deals with the asymptotic behavior of numbers of observations near extreme order statistics $X_{k:n}$ and $X_{n-r:n}$, where k and r are fixed and $n \rightarrow \infty$.

Theorem 1. *Let $l_F = -\infty$, $r_F = \infty$ and a, b, c, d be fixed positive numbers such that $\bar{\beta}(a)$, $\bar{\beta}(b)$, $\beta(c)$, $\beta(d)$ all exist and belong to $(0, 1)$. Then, for all integers $i \geq 0$, $m \geq 0$, $j = 0, 1, \dots, k-1$ and $l = 0, 1, \dots, r$,*

$$\begin{aligned} & \lim_{n \rightarrow \infty} P(K_-(n, k, a) = j, K_+(n, k, b) = i, K_-(n, n-r, c) = m, K_+(n, n-r, d) = l) \\ &= \binom{k-1}{j} [\bar{\beta}(a)]^{k-1-j} [1 - \bar{\beta}(a)]^j \binom{i+k-1}{i} [\bar{\beta}(b)]^k [1 - \bar{\beta}(b)]^i \\ & \quad \times \binom{m+r}{m} [\beta(c)]^{r+1} [1 - \beta(c)]^m \binom{r}{l} [\beta(d)]^{r-l} [1 - \beta(d)]^l \\ &= \lim_{n \rightarrow \infty} P(K_-(n, k, a) = j) \lim_{n \rightarrow \infty} P(K_+(n, k, b) = i) \lim_{n \rightarrow \infty} P(K_-(n, n-r, c) = m) \\ & \quad \times \lim_{n \rightarrow \infty} P(K_+(n, n-r, d) = l) > 0. \end{aligned} \quad (4)$$

Proof. See Dembińska and Balakrishnan (2010, Section 2). $\square$

Note that the limit in (4) factors suitably to yield the asymptotic independence of the components of the random vector in (2). Hence the following corollary follows readily from Theorem 1.

Corollary 1. *Under the assumptions of Theorem 1, the rv's $K_-(n, k, a)$, $K_+(n, k, b)$, $K_-(n, n-r, c)$ and $K_+(n, n-r, d)$ are asymptotically independent as $n \rightarrow \infty$.*

Our next goal is to establish asymptotic distributions of the random vectors $\mathbb{L}(n, k, \mathbf{a})$ and $\mathbb{M}(n, k, \mathbf{a})$ as $n \rightarrow \infty$.

Theorem 2. *Let $l_F = -\infty$ and $0 = a_0 < a_1 < \dots < a_s$ be real numbers such that $\bar{\beta}(a_i)$, $i = 1, \dots, s$, all exist and belong to $(0, 1)$. Then,*

$$\mathbb{L}(n, k, \mathbf{a}) \xrightarrow{d} \mathcal{M}ult(k-1, \bar{\beta}(a_0) - \bar{\beta}(a_1), \dots, \bar{\beta}(a_{s-1}) - \bar{\beta}(a_s)) \text{ as } n \rightarrow \infty,$$

where $\xrightarrow{d}$ stands for convergence in distribution and $\mathcal{M}ult(k-1, p_1, \dots, p_s)$ denotes the multinomial distribution with index $k-1$ and cell probabilities $p_1, \dots, p_s$.

Proof. It is known that the conditional distribution of $(X_{1:n}, \dots, X_{k-1:n})$, given $X_{k:n} = x$ is the same as unconditional distribution of order statistics of a sample $Y_1^x, \dots, Y_{k-1}^x$ from the right-truncated population with cdf $F_1^x(u) = \frac{F(u)}{F(x)}$, $u < x$. Since $L_j = \sum_{i=1}^{k-1} I_{(a_{j-1}, a_j)}(X_{k:n} - X_{i:n})$, $j = 1, \dots, s$, it yields

$$\mathbb{L}(n, k, \mathbf{a}) | X_{k:n} = x \sim \mathcal{M}ult(k-1, P(x - Y_1^x \in (a_0, a_1)), \dots, P(x - Y_1^x \in (a_{s-1}, a_s))).$$

Consequently the probability generating function (pgf) of the random vector $\mathbb{L}(n, k, \mathbf{a})$ has the form

$$\begin{aligned} g_{\mathbb{L}}(z_1, \dots, z_s) &= \mathbb{E} \left(\prod_{i=1}^s z_i^{L_i} \right) = \mathbb{E} \left[\mathbb{E} \left(\prod_{i=1}^s z_i^{L_i} | X_{k:n} \right) \right] \\ &= \mathbb{E} \left[\left(1 - \prod_{i=1}^s (1 - z_i) p_i(X_{k:n}) \right)^{k-1} \right], \end{aligned}$$

where

$$p_i(X_{k:n}) = P(X_{k:n} - Y_1^{X_{k:n}} \in (a_{i-1}, a_i) | X_{k:n}), \quad i = 1, \dots, s.$$

Note that

$$\begin{aligned} P(X_{k:n} - Y_1^{X_{k:n}} \in (a_{i-1}, a_i) | X_{k:n} = x) &= P(x - a_i < Y_1^x < x - a_{i-1}) \\ &= \frac{F(x - a_{i-1})}{F(x)} - \frac{F(x - a_i)}{F(x)} \rightarrow \bar{\beta}(a_{i-1}) - \bar{\beta}(a_i) \quad \text{as } x \rightarrow -\infty. \end{aligned}$$

Since $X_{k:n} \xrightarrow{\text{a.s.}} -\infty$ as $n \rightarrow \infty$, we get $p_i(X_{k:n}) \xrightarrow{\text{a.s.}} \bar{\beta}(a_{i-1}) - \bar{\beta}(a_i)$ as $n \rightarrow \infty$. Hence the Bounded Convergence Theorem yields

$$g_{\mathbb{L}}(z_1, \dots, z_s) \xrightarrow{x \rightarrow -\infty} \left(1 - \prod_{i=1}^s (1 - z_i) (\bar{\beta}(a_{i-1}) - \bar{\beta}(a_i)) \right)^{k-1} \quad \text{for } z_i \in [0, 1],$$

and the application of the continuity theorem for pgf's finishes the proof. $\square$

Theorem 3. *Under the assumptions of Theorem 2,*

$$\mathbb{M}(n, k, \mathbf{a}) \xrightarrow{d} \mathcal{N}Mult(k, p_1, \dots, p_s) \quad \text{as } n \rightarrow \infty,$$

where $\mathcal{N}Mult(k, p_1, \dots, p_s)$ denotes the negative multinomial distribution with index k and cell probabilities $p_1, \dots, p_s$ such that

$$p_j = \bar{\beta}(a_s) \left\{ \frac{1}{\bar{\beta}(a_j)} - \frac{1}{\bar{\beta}(a_{j-1})} \right\}, \quad j = 1, \dots, s.$$

Proof. See Dembińska and Iliopoulos (2010). $\square$

3 Observations near central order statistics

In this section, we consider the asymptotic properties of numbers of observations that fall into the left and right vicinity of a central order statistic $X_{k_n:n}$, where $\lim_{n \rightarrow \infty} k_n/n = \alpha \in (0, 1)$. As mentioned in Introduction, if F is continuous and strictly increasing in a neighborhood of the α -th quantile, then for any fixed $a > 0$,

$$K_-(n, k_n, a) \xrightarrow{P} \infty, \quad K_+(n, k_n, a) \xrightarrow{P} \infty.$$

Therefore, in the case of central order statistics, in order to obtain nondegenerate limit laws for the rv's $K_-(n, k_n, a)$ and $K_+(n, k_n, a)$, one has to allow a to vary with n . For example, for special choices of a_n , one gets a Poisson limit law for $K_-(n, k_n, a)$ and $K_+(n, k_n, a)$.

Theorem 4 (Dembńska, Stepanov and Wesołowski, 2007). *Let $\alpha \in (0, 1)$ and there exist exactly one γ such that $F(\gamma) = \alpha$. Suppose that*

$$a_n = \gamma - F^{\leftarrow}(\alpha - \lambda/n), \quad n = 1, 2, \dots \text{ for some } \lambda > 0$$

and

$$\lim_{(x,y) \rightarrow (\gamma, 0^+)} \frac{F(x) - F(x-y)}{F(\gamma) - F(\gamma-y)} = 1. \quad (5)$$

Then,

$$K_-(n, k_n, a_n) \xrightarrow{d} \mathcal{P}(\lambda),$$

where $\mathcal{P}(\lambda)$ denotes the Poisson distribution with mean λ , and $F^{\leftarrow}(x) := \inf\{s : F(s) \geq x\}$, $x \in (0, 1)$. Furthermore, if

$$b_n = F^{\leftarrow}(\alpha + \lambda/n) - \gamma, \quad n = 1, 2, \dots \text{ for some } \lambda > 0$$

and

$$\lim_{(x,y) \rightarrow (\gamma, 0^+)} \frac{F(x+y) - F(x)}{F(\gamma+y) - F(\gamma)} = 1, \quad (6)$$

we have

$$K_+(n, k_n, b_n) \xrightarrow{d} \mathcal{P}(\lambda).$$

The following theorem shows that, under regularity conditions of the form (5) and (6) and for appropriately chosen sequences $\{a_n^{(i)}, n \geq 1\}$ and $\{b_n^{(i)}, n \geq 1\}$, $i = 1, \dots, s$, the numbers of observations that fall into the left $a_n^{(i)}$ -vicinities and right $b_n^{(i)}$ -vicinities of central order statistics $X_{k_n^{(i)}:n}$, $i = 1, \dots, s$, are asymptotically independent.

Theorem 5. *Let $\{k_n^{(i)}, n \geq 1\}$, $i = 1, \dots, s$, be sequences of positive integers such that $\lim_{n \rightarrow \infty} k_n^{(i)}/n = \alpha_i \in (0, 1)$, $i = 1, \dots, s$, and the α_i 's are distinct. Assume that for $i = 1, \dots, s$ there exists γ_i such that $F(\gamma_i) = \alpha_i$ and F is strictly increasing in a*

neighborhood of γ_i . If (5) and (6) hold with $\gamma = \gamma_i$, $i = 1, \dots, s$, and for some $\lambda_{ij} > 0$, $i = 1, \dots, s$, $j = 1, 2$,

$$a_n^{(i)} = \gamma_i - F^{\leftarrow}(\alpha_i - \lambda_{1i}/n), \quad b_n^{(i)} = F^{\leftarrow}(\alpha_i + \lambda_{2i}/n) - \gamma_i, \quad n = 1, 2, \dots,$$

then the rv's $K_-(n, k_n^{(1)}, a_n^{(1)})$, $K_+(n, k_n^{(1)}, b_n^{(1)})$, $\dots$, $K_-(n, k_n^{(s)}, a_n^{(s)})$, $K_+(n, k_n^{(s)}, b_n^{(s)})$ are asymptotically independent, and moreover

$$K_-(n, k_n^{(i)}, a_n^{(i)}) \xrightarrow{d} \mathcal{P}(\lambda_{1i}), \quad K_+(n, k_n^{(i)}, b_n^{(i)}) \xrightarrow{d} \mathcal{P}(\lambda_{2i}), \quad i = 1, \dots, s.$$

Proof. See Dembińska and Balakrishnan (2010, Section 3). $\square$

4 Randomly indexed samples

Now let us consider randomly indexed samples $X_1, X_2, \dots, X_{N(t)}$, where $(N(t), t \geq 0)$ is a process of positive integer-valued rv's independent of $(X_n, n \geq 1)$. Define

$$\mathcal{K}_-(t, k, a) = \begin{cases} \#\{j \in \{1, \dots, N(t)\} : X_j \in (X_{k:N(t)} - a, X_{k:N(t)})\} & \text{if } N(t) \geq k \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathcal{K}_+(t, k, a) = \begin{cases} \#\{j \in \{1, \dots, N(t)\} : X_j \in (X_{k:N(t)}, X_{k:N(t)} + a)\} & \text{if } N(t) \geq k \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathcal{L}(t, k, \mathbf{a}) = \begin{cases} \mathbb{L}(N(t), k, \mathbf{a}) & \text{if } N(t) \geq k \\ 0 & \text{otherwise} \end{cases}, \quad \mathcal{M}(t, k, \mathbf{a}) = \begin{cases} \mathbb{M}(N(t), k, \mathbf{a}) & \text{if } N(t) \geq k \\ 0 & \text{otherwise} \end{cases}.$$

Randomly indexed samples occur in a natural way when one consider a model of insurance claims in which the size of the n th claim is X_n and claims occur at the event times of the non-decreasing process $(N(t), t \geq 0)$. Then $X_{N(t):N(t)}$ is the largest claim size up to time t and $\mathcal{K}_-(t, N(t), a)$ describes the number of near-maximum insurance claims. Basic distributional results as well as limit laws for the rv $\mathcal{K}_-(t, N(t), a)$ were given by Li and Pakes (2001). Some extensions of their results to the case of not necessarily iid claim sizes $(X_n, n \geq 1)$ and to the bivariate claims $(\mathbb{X}_n = (X_{n1}, X_{n2}), n \geq 1)$ can be found in Hashorva (2003, 2004).

Here we show that all theorems of previous sections of this paper can be generalized to the case of randomly indexed samples. This is due to the following theorem given by Dembińska (2010, Theorem 2.1; see also Lemma 2.5.6 of Embrechts, Klüppelberg and Mikosch, 1997, for a univariate version of this theorem.)

Theorem 6. Let $(\mathbb{Y}_n = (Y_n^{(1)}, \dots, Y_n^{(r)}), n \geq 1)$ be a sequence of r -dimensional random vectors and $(N(t), t \geq 0)$ be a process of positive integer-valued rv's. Suppose that $(\mathbb{Y}_n, n \geq 1)$ and $(N(t), t \geq 0)$ are independent, $N(t) \xrightarrow{P} \infty$ as $t \rightarrow \infty$, and $\mathbb{Y}_n \xrightarrow{d} \mathbb{Y} = (Y_1, \dots, Y_r)$ as $n \rightarrow \infty$. Then $\mathbb{Y}_{N(t)} \xrightarrow{d} \mathbb{Y}$ as $t \rightarrow \infty$.

For example, Theorems 1, 2, 3 and 6 immediately imply the following results.

Theorem 7. Let $(N(t), t \geq 0)$ be a process of positive integer-valued rv's independent of $(X_n, n \geq 1)$. Suppose that $N(t) \xrightarrow{P} \infty$ as $t \rightarrow \infty$ and the assumptions of Theorem 2 hold. Then the random vectors

- $\mathbb{L}(n, k, \mathbf{a})$ and $\mathcal{L}(t, k, \mathbf{a})$,
- $\mathbb{M}(n, k, \mathbf{a})$ and $\mathcal{M}(t, k, \mathbf{a})$

have the same limiting law.

Theorem 8. Under the assumptions of Theorem 7 with “assumptions of Theorem 2” replaced by “assumptions of Theorem 1”, the rv's $\mathcal{K}_-(t, k, a)$, $\mathcal{K}_+(t, k, b)$, $\mathcal{K}_-(t, N(t) - r, c)$ and $\mathcal{K}_+(t, N(t) - r, d)$ are asymptotically independent.

Similarly, Theorems 5 and 6 show under which conditions the numbers of observations falling into the left and right neighborhoods of central randomly indexed order statistics are asymptotically independent.

Theorem 9. Let $(N(t), t \geq 0)$ be a process of positive integer-valued rv's independent of $(X_n, n \geq 1)$ and such that $N(t) \xrightarrow{P} \infty$ as $t \rightarrow \infty$. Then, under the assumptions of Theorem 5, the rv's $\mathcal{K}_-(t, k_{N(t)}^{(1)}, a_{N(t)}^{(1)})$, $\mathcal{K}_+(t, k_{N(t)}^{(1)}, b_{N(t)}^{(1)})$, $\dots$, $\mathcal{K}_-(t, k_{N(t)}^{(s)}, a_{N(t)}^{(s)})$, $\mathcal{K}_+(t, k_{N(t)}^{(s)}, b_{N(t)}^{(s)})$ are asymptotically independent as $t \rightarrow \infty$, and moreover

$$\mathcal{K}_-(t, k_{N(t)}^{(i)}, a_{N(t)}^{(i)}) \xrightarrow{d} \mathcal{P}(\lambda_{1i}), \mathcal{K}_+(t, k_{N(t)}^{(i)}, b_{N(t)}^{(i)}) \xrightarrow{d} \mathcal{P}(\lambda_{2i}), i = 1, \dots, s \text{ as } t \rightarrow \infty.$$

The following result presents how the limiting distribution given in Theorem 9 will change if we use nonrandom sequences $(a_n, n \geq 1)$ and $(b_n, n \geq 1)$ instead of $(a_{N(t)}, t \geq 0)$ and $(b_{N(t)}, t \geq 0)$.

Theorem 10. Let $(N(t), t \geq 0)$ be a process of positive integer-valued rv's independent of $(X_n, n \geq 1)$ and such that

$$\frac{N(t)}{t} \xrightarrow{P} Z \text{ as } t \rightarrow \infty, \text{ where } Z \text{ is a non-negative rv.} \quad (7)$$

Then, under the assumptions of Theorem 5, we have

$$\begin{aligned} & \lim_{n \rightarrow \infty} P \left(\mathcal{K}_-(n, k_{N(n)}^{(i)}, a_n^{(i)}) = j_i, \mathcal{K}_+(n, k_{N(n)}^{(i)}, b_n^{(i)}) = m_i, i = 1, \dots, s \right) \\ &= E \left(\prod_{i=1}^s e^{-\lambda_{1,i}Z} \frac{(\lambda_{1,i}Z)^{j_i}}{j_i!} \times \prod_{i=1}^s e^{-\lambda_{2,i}Z} \frac{(\lambda_{2,i}Z)^{m_i}}{m_i!} \right). \end{aligned} \quad (8)$$

Proof. See Dembińska (2010, Section 3). $\square$

For practical purposes, it often suffices to consider processes $(N(t), t \geq 0)$ satisfying a stronger condition than (7):

$$\frac{N(t)}{t} \xrightarrow{P} \tau, \text{ as } t \rightarrow \infty, \text{ for some constant } \tau > 0. \quad (9)$$

Condition (9) is satisfied, for example, by the homogeneous Poisson process with intensity τ , and more generally, by the renewal counting processes with inter-arrival times being iid rv's with finite expectations; see Embrechts, Klüppelberg and Mikosch (1997, Section 2.5.2).

If in Theorem 10 we replace (7) with the stronger condition (9) than the limit in (8) will factor suitably to yield the asymptotic independence of $\mathcal{K}_-(n, k_{N(n)}^{(1)}, a_n^{(1)})$, $\mathcal{K}_+(n, k_{N(n)}^{(1)}, b_n^{(1)})$, $\dots$, $\mathcal{K}_-(n, k_{N(n)}^{(s)}, a_n^{(s)})$ and $\mathcal{K}_+(n, k_{N(n)}^{(s)}, b_n^{(s)})$.

Theorem 11. *Let $(N(t), t \geq 0)$ be a process of positive integer-valued rv's independent of $(X_n, n \geq 1)$ and satisfying (9). Then, under the assumptions of Theorem 5, the rv's $\mathcal{K}_-(n, k_{N(n)}^{(1)}, a_n^{(1)})$, $\mathcal{K}_+(n, k_{N(n)}^{(1)}, b_n^{(1)})$, $\dots$, $\mathcal{K}_-(n, k_{N(n)}^{(s)}, a_n^{(s)})$, $\mathcal{K}_+(n, k_{N(n)}^{(s)}, b_n^{(s)})$ are asymptotically independent. Moreover, for $i = 1, \dots, s$,*

$$\mathcal{K}_-(n, k_{N(n)}^{(i)}, a_n^{(i)}) \xrightarrow{d} \mathcal{P}(\lambda_{1i}\tau) \text{ and } \mathcal{K}_+(n, k_{N(n)}^{(i)}, b_n^{(i)}) \xrightarrow{d} \mathcal{P}(\lambda_{2i}\tau), \text{ as } n \rightarrow \infty.$$

References

1. Balakrishnan, N., Stepanov, A.: A note on the number of observations near an order statistic. J. Statist. Plann. Inf. **134**, 1–14 (2005)
2. Dembińska, A.: On numbers of observations near randomly indexed order statistics. Statist. Probab. Lett. **80**, 309–317 (2010)
3. Dembińska, A., Balakrishnan, N.: The asymptotic distribution of numbers of observations near order statistics. J. Statist. Plann. Inf. **138**, 2552–2562 (2008)
4. Dembińska, A., Balakrishnan, N.: On the asymptotic independence of numbers of observations near order statistics. Statistics (2010) (to appear).
5. Dembińska, A., Iliopoulos, G.: On the asymptotics of numbers of observations in random regions determined by order statistics. Manuscript (2010)
6. Dembińska, A., Stepanov, A., Wesołowski, J.: How many observations fall in a neighborhood of an order statistic? Comm. Statist. – Theory & Methods **36**, 851–867 (2007)
7. Embrechts, P., Klüppelberg C., Mikosch T.: Modelling Extremal Events for Insurance and Finance. Springer-Verlag, Berlin (1997)
8. Hashorva, E.: On the number of near-maximum insurance claim under dependence. Insurance Math. Econom. **32**, 37–49 (2003)
9. Hashorva, E.: Bivariate maximum insurance claim and related point processes. Statist. Probab. Lett. **69**, 117–128 (2004)
10. Hu, Z., Su, C.: Limit theorems for the number and sum of near-maxima for medium tails. Statist. Probab. Lett. **63**, 229–237 (2003)
11. Li, Y.: A note on the number of records near the maximum. Statist. Probab. Lett. **43**, 153–158 (1999)
12. Li, Y., Pakes, A.: On the number of near-maximum insurance claims. Insurance: Mathematics and Economics **28**, 309–323 (2001)
13. Pakes, A.: The number and sum of near-maxima for thin-tailed populations. Adv. Appl. Probab. **32**, 1100–1116 (2000)
14. Pakes, A., Li, Y.: Limit laws for the number of near maxima via the Poisson approximation. Statist. Probab. Lett. **40**, 395–401 (1998)
15. Pakes, A., Steutel, F.W.: On the number of records near the maximum. Austral. J. Statist. **39**, 179–193 (1997)