

An analysis of methods to make consistent the binary matrices comparisons

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Abstract This paper presents an empirical analysis of the response of several methods proposed in literature to construct indices for consistent multiple comparisons. To evaluate the various scaling methods, we have used official data furnished by Istat, and not simulations as is usually in study in this kind. The most important results that we have achieved are the close concordance of the weights obtained with the various methods and the robustness of the evaluations performed.

Keywords: Ratio-scale matrices; Multiple comparisons; Index numbers.

1 Introduction

Our purpose is to examine some procedures of economic entity comparisons. Such comparisons may be bilateral (or binary), when individual pairs of entities are compared, or they may be multiple (or multilateral) when the intention is instead to grade the entities compared. As is evident and also shown by a large body of literature, binary comparisons are simpler to set up and easier to implement. However, the information obtained in this way and arranged in appropriate ratio-scale matrices is normally inconsistent when used for multilateral comparisons. We shall illustrate this aspect in the second section, where we consider matrices relative to binary comparisons of preference judgements and survey the techniques with which such ratio-scale matrices can be adjusted so that they are consistent. Afterwards we discuss methods

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which enable consistent matrices to be constructed even independently of matrices for binary relationships. This is a line of inquiry which has been pursued and developed especially by economic statisticians. The results obtained from the application of the methods discussed in the second sections, results which enable the entities compared to be ranked consistently are analysed in the third section.

2 Methods to give consistency the binary comparisons

Binary comparisons between alternatives in order to rank preferences for each of them, or in other words, to determine an ordered set of weights to be associated with them, have for some time been the subject of a broad strand of studies [5] falling under the general heading of AHP (Analytic Hierarchy Process) and which is still developing in various directions. Of particular interest are those involving the more systematic use of statistical tools [1, 3] and those that combine AHP with various methods of multicriterion analysis [11, 12]. To restrict the discussion to the matters treated empirically in this study, we consider the set $A = \{A_1, \dots, A_n\}$ of the n alternatives to compare, and let $\mathbf{C}$ an order square n matrix whose generic element c_{ij} expresses the extent to which alternative A_i is more important than A_j , assuming that this extent is expressible on a ratio scale measurement. At most $n(n-1)/2$ comparisons are required because a minimum consistency is imposed whereby:

$$c_{ij} = 1/c_{ji} \quad i, j = 1..n \quad (1)$$

so that if, for example $c_{ij}=3$ (A_i is deemed to be three times more important than A_j), the importance of A_j is one third that of A_i , hence $c_{ji}=1/3$. It straightforwardly follows from (1) that $c_{ii} = 1$ for $i=1..n$. Matrices with positive elements possessing property (1) are known as “positive reciprocal matrices”. It should be pointed out, however, that constraint (1) does not ensure complete consistency when multiple comparisons are made. In fact, this consistency only comes about if the elements of the positive reciprocal matrix $\mathbf{C}$ satisfy the relations:

$$c_{ij} = c_{ik}c_{kj} \quad \forall i, j, k \quad (2)$$

If the weights (importance) w_i to associate with the individual alternatives A_i were known, and if matrix $\mathbf{C}$ were consistent, one would have:

$$c_{ij} = w_i/w_j \quad i, j = 1..n \quad (3)$$

Note that the weights w_i , which we shall consider in what follows as components of the vector $\mathbf{w}$, are determined up to a multiplication by a constant, which is the arbitrarily determinable unit measuring the importance of the alternatives. As it is evident from (3), in the case of perfect consistency, the vector $\mathbf{w}$ of the weights would be immediately deducible from any row or column of matrix $\mathbf{C}$. In practice, however, only rarely this matrix display the complete consistency defined by (2) when it is actually constructed.

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This feature has prompted the development of various methods to determine the weights by minimizing the divergence, opportunely defined, between the values c_{ij} and the theoretical ones w_i/w_j .

Various studies [6,10] have used stochastic simulation techniques to assess the performance of the various adjustment methods. In our empirical analysis, we applied the best known and most widely used of these methods, which are now briefly surveyed.

Before beginning the survey, however, we would point out that the multiplier constant up to which the weights are defined can be chosen in various ways: by giving unitary value to one component of vector $\mathbf{w}$, imposing the unit-sum constraint or the unitary product constraint on the components of vector $\mathbf{w}$. Obviously, this is a arbitrary choice made purely for reasons of computational convenience. Referring to the literature for the details following the considered methods are reassumed:

1) DE - *Dominant eigenvalue method* [14]: if the relations in (3) hold, then matrix $\mathbf{C}$ has one single eigenvalue different from zero whose value corresponds to the order of the matrix, and moreover the vector of the weights $\mathbf{w}$ coincides with the eigenvector associated with the dominant eigenvalue λ :

$$\mathbf{C}\mathbf{w} = \lambda\mathbf{w} \quad (4)$$

$\mathbf{w}$ is considered to be a reasonable evaluation of the vector of the weights. Frobenius's theorem ensures the positivity of the components of $\mathbf{w}$ and therefore the latter's acceptability in the context considered.

2) MDE - *Modified dominant eigenvalue method* [4]: this method is a variant of the previous one. Bearing in mind that matrix $\mathbf{C}$ is reciprocal, this technique considers only the upper triangle. Using $\mathbf{T}$ to denote the matrix such that $t_{ij}=c_{ij}$ if $i \geq j$ and $t_{ij}=0$ otherwise and $\mathbf{G}$ to denote the diagonal matrix with the elements $g_{ii}=n-i+1$ for $i=1..n$, the weights correspond to the solution of the system: $(\mathbf{G}^{-1}\mathbf{T} - \mathbf{U})\mathbf{w} = 0$. Where $\mathbf{U}$ is a unitary matrix. The solution is obtained recursively by means of the relations:

$$w_i = \frac{\sum_{j=i+1}^n c_{ij} w_j}{(n-1)} \quad i = 1..n-1 \quad (5)$$

3) DLS - *Direct least squares method* [15]: The weights are determined so as to minimize the following objective functions. In this case the solution $\mathbf{w}$ does not have an analytical definable expression but must be calculated with iterative methods.

$$\Phi_1(\mathbf{w}) = \sum_{i=1}^n \sum_{j=1}^n \left(c_{ij} - \frac{w_i}{w_j} \right)^2 \quad (6)$$

4) LLS - *Logarithmic least squares method* [15]: Given the multiplicative nature of the positive reciprocal matrix $\mathbf{C}$, see the (2), it is natural to measure the divergences between observed values c_{ij} and the theoretical values w_i/w_j considering the ratio between them, that is, by calculating the relative error given by the deviation of this ratio from unity:

$$\Phi_3(\mathbf{w}) = \sum_{i=1}^n \sum_{j=1}^n \left(\ln \frac{c_{ij}}{w_i / w_j} \right)^2 = \sum_{i=1}^n \sum_{j=1}^n (\ln c_{ij} - \ln w_i - \ln w_j)^2 \quad (7)$$

Selecting the arbitrary multiplying constant of the weights so that their product is unitary, the components of vector $\mathbf{w}$ that minimize (7) correspond to the geometric mean of the elements in the corresponding row of matrix $\mathbf{C}$:

$$w_i = \left(\prod_{j=1}^n c_{ij} \right)^{1/n} \quad i=1..n \quad (8)$$

5) WLS - *Weighted least squares method* [2]: This is a method similar to the previous. In this case the objective function to minimize takes the following form:

$$\Phi_2(\mathbf{w}) = \sum_{i=1}^n \sum_{j=1}^n \left(c_{ij} - \frac{w_i}{w_j} \right)^2 w_j^2 = \sum_{i=1}^n \sum_{j=1}^n (c_{ij} w_j - w_i)^2 \quad (9)$$

It has been shown that unlike the previous DLS method, which may have multiple solutions, (9) admits to only one and strictly positive solution. This can be straightforwardly obtained by solving the following system of linear equations:

$$\left(\sum_{i \neq k}^n c_{ik}^2 + n - 1 \right) w_k - \sum_{j \neq k}^n (c_{jk} + c_{kj}) w_j + \lambda = 0 \quad k=1..n \quad (10)$$

generated by the first-order conditions for the Lagrange auxiliary function constructed by taking (9) as the objective function and constraining the weights to unitary sum.

The methodology for performing accurate comparisons among economic phenomena observed on different occasions (namely index number theory) is a key area of research for economic statisticians, and to which they have made a number of crucial contributions. With index numbers, comparison is made among the relative differences of a phenomenon by examining their corresponding intensities on different occasions.

For example, if we consider the unit price of a generic good on two occasions A and B , which may be two regions, two periods, and so on, the ratios p_A/p_B and p_B/p_A measure the relative difference in the price on occasion A with respect to occasion B considered as the base situation, and vice versa. Simple index numbers obviously satisfy the consistency conditions mentioned in the second section. However, when the phenomenon is analysed across a number of entities, or, to refer to the example again, when the unit prices of n goods are considered, the statistical relationships that simultaneously and succinctly measure the relative diversity of prices in the situations compared are known as ‘complex index numbers’. There are various methods with which to define complex index numbers. For details see [13]. However, many of the complex indexes proposed and widely used do not permit consistent comparisons. The most frequently used indices, Laspeyres and Paasche indices, do not satisfy even the minimum consistency conditions (1) of reversibility of situations. This is a problem with particular practical implications for the calculation of purchasing power parities.

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The theoretical and practical importance of comparison among complex index numbers is certainly not restricted to price index numbers, for there is an immediate logical symmetry between the latter and the index numbers of quantities. Nevertheless, in what follows, we shall restrict our treatment to the former, both because it was on these that we conducted our empirical analysis and because they suffice to illustrate the issue analysed here.

In the following we briefly survey the best known methods for the construction of complex and consistent index numbers. We assume that n situations are to be compared, each of them characterized by m goods described by the following vectors of prices and quantities:

$$\mathbf{p}_i = [p_{i1}, p_{i2}, \dots, p_{im}]; \quad \mathbf{q}_i = [q_{i1}, q_{i2}, \dots, q_{im}] \quad i=1..n \quad (11)$$

1) GEKS - *The Gini-Eltető-Köves-Szulc method* [9]: This method is based on the matrix $\mathbf{C}$ of binary comparisons whose elements are Fisher price index numbers:

$$c_{ij} = \left[\left(\frac{\sum_{k=1}^m p_{ik} q_{jk}}{\sum_{k=1}^m p_{ik} q_{ik}} \right) \left(\frac{\sum_{k=1}^m p_{jk} q_{jk}}{\sum_{k=1}^m p_{ik} q_{jk}} \right) \right]^{0.5} \quad i, j=1..n \quad (12)$$

The LLS method mentioned earlier has been proposed in order to give consistency to the comparisons expressed by the $\mathbf{C}$ matrix which is evidently positive reciprocal.

2) T - *The Theil method* [16]: In this case, too, consideration is made of a $\mathbf{C}$ matrix of binary comparisons, which are then made consistent by means of the logarithmic least squares method. Unlike in the previous case, however, the elements that make up the matrix are Törnquist bilateral indices:

$$c_{ij} = \prod_{k=1}^m \left(\frac{p_{ik}}{p_{jk}} \right)^{0.5 \left[\left(\frac{p_{ik} q_{ik}}{\sum_{k=1}^m p_{ik} q_{ik}} \right) + \left(\frac{p_{jk} q_{jk}}{\sum_{k=1}^m p_{jk} q_{jk}} \right) \right]} \quad (13)$$

3) ECLA - *The Economic Commission for Latin America method* [9]: Its distinctive feature is that it directly and consistently constructs the $\mathbf{C}$ matrix of comparisons, whose elements are defined in the following manner:

$$c_{ij} = \frac{\sum_{k=1}^m p_{ik} \bar{q}_k}{\sum_{k=1}^m p_{jk} \bar{q}_k} \quad i, j=1..n \quad \text{where} \quad \bar{q}_k = \frac{1}{n} \sum_{i=1}^n q_{ik} \quad (14)$$

It is evident that any linear combination of the quantities $\bar{q}_k = \sum_{i=1}^n \alpha_i q_{ik}$, with positive weights (α_i), gives rise to consistent comparisons.

4) GK - *The Geary-Khamis method* [7]: Nor does this method require the construction of a matrix of binary comparisons to be then rendered consistent. Rather, it is constructed directly, by means of (3), from the weights w_i , which act as conversion factors. This technique is of iterative. Each iteration involves two phases: in the first, after giving w_i an arbitrary initial value, (15) is used to determine the mean prices (π_k) of each good in the basket of the statistical collective. In the second, the latter is

inserted in (16) to determine the conversion factors w_i (weights). The procedure is reiterated until the solutions of two successive iterations are judged equal to each other.

$$\pi_k = \frac{\sum_{i=1}^n w_i p_{ik} q_{ik}}{\sum_{i=1}^n q_{ik}} \quad k=1..m \quad (15)$$

$$w_i = \frac{\sum_{k=1}^m \pi_k q_{ik}}{\sum_{k=1}^m p_{ik} q_{ik}} \quad i=1..n \quad (16)$$

5) *G - The Gerardi method* [8]: This technique does not substantially differ from the GK method. The only difference being that in this case the mean prices of the individual goods are calculated using a geometric mean of the prices observed on the n occasions:

$$\pi_k = \left(\prod_{i=1}^n p_{ik} \right)^{1/n} \quad k=1..m \quad (17)$$

3 Empirical analysis of the methods to construct indices for multiple comparisons

In this section we describe the series of experiments conducted on empirical data from official sources (ISTAT) in order to examine the responses of the various methods surveyed above. As a synthesis measure we consider the following consistency index (Mean Squared Deviation) of the weights with respect to those calculated as a mean of the nine methods applied:

$$MSD_b = \left[\frac{1}{n} \sum_{i=1}^n (w_{ib} - \bar{w}_i)^2 \right]^{1/2}; \quad b=1..9; \quad \bar{w}_i = \frac{1}{9} \sum_{b=1}^9 w_{ib} \quad i=1..n \quad (18)$$

In detail, the analysis considered the annual statistics on Italian industrial production and manufacturing and cover the period from 1994 to 1998. It should also be pointed out that the data series are not perfectly comparable, and that, in view of the purpose of our inquiry, we decided to eliminate all items that: were not present in all years, were not published in some years because they were covered by statistical confidentiality, were not distinguished by prices and quantities sold and had been surveyed over the years using different measurement systems and therefore could not be related to the same unit of measurement. This screening procedure led to the exclusion of three sectors. For each of the remained economic sector, two distinct binary comparison matrices were constructed: the Fisher price index numbers and the Törnqvist price index numbers. Since these matrices are extremely similar among them we have considered only the matrix of Fisher index numbers. For each year this matrix, using

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the nine methods previously considered, has been made consistent. The results of these analyses, which as normalized to the first year considered, are set out for the first economic sector in Table 1, and synthesize (the results in detail are available from the authors) through the MSD consistency index, in Table 2 for the remainders sectors.

Table 1: Sector 1 - production and sale of non-energy-producing minerals.

	DE	MDE	DLS	WLS	LLS GEKS	GK	G	T	ECLA	Mean
1994	1	1	1	1	1	1	1	1	1	1
1995	0.8860	0.8859	0.8859	0.8860	0.8860	0.9026	0.8984	0.8857	0.8676	0.8871
1996	0.9211	0.9265	0.9210	0.9213	0.9211	0.9128	0.9207	0.9085	0.9360	0.9210
1997	0.9922	0.9916	0.9921	0.9925	0.9922	0.9310	0.9944	0.9617	1.0241	0.9858
1998	1.0105	1.0051	1.0105	1.0107	1.0105	0.9495	1.0124	0.9827	1.0420	1.0038
MSD	0.0042	0.0036	0.0041	0.0043	0.0042	0.0353	0.0074	0.0153	0.0266	

Table 2: MSD consistency index by economics sectors and methods.

Others economics sectors.										
LLS										
Production and sale of:	DE	MDE	DLS	WLS	GEKS	GK	G	T	ECLA	
2) food and beverages	0.13	0.16	0.13	0.14	0.13	1.08	0.17	0.35	0.89	
3) textiles	0.69	0.59	0.73	0.79	0.69	6.75	2.76	1.86	7.93	
4) garments and furs	0.53	0.57	0.64	0.68	0.53	5.46	1.91	1.23	5.78	
5) leather and leather goods	0.58	0.55	0.63	0.65	0.58	9.12	3.09	1.32	10.02	
6) wood and wooden products	0.06	0.1	0.07	0.06	0.07	0.51	0.33	0.26	0.77	
7) paper-making, pulp, paper	0.25	0.25	0.25	0.25	0.25	1.4	0.61	0.33	1.06	
8) printed and recorded prod.	0.18	0.36	0.19	0.18	0.17	1.12	1.35	0.21	1.7	
9) chemic. and man-made fibres	0.07	0.07	0.07	0.08	0.07	1.15	0.58	0.1	1.32	
10) rubber and plastic articles	1.00	1.01	1.00	1.00	1.00	5.61	4.59	2.02	7.19	
11) non-metallic mineral prod.	3.62	3.62	3.62	3.62	3.62	5.85	3.11	7.19	1.97	
12) metals and alloys	0.49	0.48	0.49	0.50	0.49	2.82	0.99	0.8	2.09	
13) machinery and equipment	0.62	0.81	0.64	0.52	0.62	4.73	1.79	1.22	5.12	
14) office machinery, computers	0.37	0.69	0.37	0.37	0.37	2.19	1.78	0.94	1.88	
15) electric. mach. and applian.	0.45	0.74	0.46	0.44	0.45	5.69	3.60	2.25	9.52	
16) radio and televis. apparatus	0.84	0.88	1.14	1.27	0.86	2.81	4.33	3.26	2.79	
17) surgical and orthop. prod.	1.07	1.12	1.04	1.02	1.08	6.4	5.59	1.83	10.72	
18) vehicles, trail. and engines	0.05	0.10	0.05	0.05	0.05	0.42	0.25	0.06	0.42	
19) other means of transport	1.66	1.65	1.67	1.66	1.65	6.78	3.30	4.45	6.88	
20) furniture and oth. manufac.	0.41	0.57	0.41	0.42	0.41	2.99	1.34	1.12	3.15	

N. B. all value are multiplied by 100

The most important result yielded by this large series of calculations, and the one most immediately grasped, is the close conformity of the weight values obtained with the various methods. This feature is also highlighted by the mean squared deviation (MSD), which in the majority of cases has extremely low values. In particular the ECLA, GK and K methods display values which for same sectors are slightly different from those of the other methods, whilst in other situations (sectors: 5, 17 and 19) these differences are more substantial. All in all, the analysis conducted show the high degree of concordance among the weight.

4 Conclusions

There is a large body of literature on the methodology with which to perform consistent comparisons. These methods have been analysed in independent manner by scholars of decision-making processes in business, and by economic statisticians with regard to the comparison of economic aggregates. In this paper we have emphasised the close connection between the two approaches. Then we have conducted a series of calculations using time series data. The results that we believe to be most significant are the close concordance, sometimes almost the coincidence, of the weights obtained with the various methods and the notable robustness of the evaluations performed. The most immediate conclusion to be drawn from our analysis is that these methods should be used more widely. Which method in particular should be selected is of little importance from the practical point of view, given the extremely close concordance obtained in the responses.

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