

A Consistent Least Squares Estimator for the Simultaneous Autoregressive Model on the Planar Lattice¹

Uno Stimatore dei Minimi Quadrati Consistente per il Modello Simultaneo Autoregressivo sul Reticolo Piano

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Riassunto: E' noto che il metodo dei minimi quadrati ordinari fornisce stimatori inconsistenti nel caso del modello spaziale autoregressivo simultaneo (SAR). Al fine di ottenere degli stimatori consistenti, nel lavoro si sfrutta l'equivalenza del modello SAR di ordine p ad un modello spaziale autoregressivo condizionato (CAR) di ordine superiore, per cui gli stimatori OLS sono consistenti. Imponendo che i parametri del modello CAR soddisfino le restrizioni non lineari derivanti dal modello SAR sottostante, si perviene ad un stimatore dei minimi quadrati non lineari. Tale stimatore è consistente e risulta agevole da computare anche nel caso di reticoli di grandi dimensioni. Il lavoro si chiude con uno studio Monte Carlo volto a valutare la performance comparativa dello stimatore proposto e dello stimatore di massima verosimiglianza in campioni finiti.

Keywords: SAR model, CAR model, Pseudo-likelihood, Non linear least squares.

1. Introduction

Due to the correlation between explanatory variables and the error term, OLS estimators of the simultaneous spatial autoregressive (SAR) model parameters are well known to be biased and inconsistent (Whittle, 1954). On the contrary, in the case of the conditional spatial autoregressive (CAR) model, residuals are orthogonal to regressors by construction, and OLS estimators are consistent (Besag, 1974). This feature suggests the possibility of exploiting the equivalence of SAR models to higher order CAR models (see, e.g., Cliff and Ord, 1981, Whittle, 1986) to estimate the SAR parameters by means of the equivalent CAR specification. By a proper definition of model order p , Whittle (1986) proves that to each $SAR(p)$ specification corresponds a $CAR(2p)$ specification whose parameters will be non linear functions of the $SAR(p)$ model parameters. In the paper the $SAR(p)$ model parameters are estimated by applying non linear least squares (NLLS) to the corresponding constrained $CAR(2p)$ model. The NLLS estimator can be computed iteratively by performing a sequence of simple OLS regressions on the linearized model, and, differently from maximum likelihood, is computationally inexpensive, thus being particularly useful for fitting SAR models to large data sets, like the ones used in digital image analysis.

2. The least squares estimator

The SAR model on the planar lattice is defined by the following stochastic difference equation

¹ The views expressed in the paper are those of the author's, and do not necessarily reflect those of the Banca d'Italia.

$$y_{ij} = \sum_{h=-l}^l \sum_{k=-m}^m f_{hk} y_{i+h, j+k} + u_{ij}$$

where $f_{00} = 0$, $E(u_{ij}) = 0$, $E(u_{ij}u_{i+h, j+k}) = s^2$ if $h=k=0$ and $E(u_{ij}u_{i+h, j+k}) = 0$ elsewhere.

Introducing the row and column displacement operators L_r and L_c , defined as $L_r^h L_c^k y_{ij} = y_{i-h, j-k}$, and setting $\mathbf{L} = [L_c, L_r]'$, model's expression becomes

$$f(\mathbf{L})y_{ij} = u_{ij}, \quad \text{with } f(\mathbf{L}) = 1 - \sum_{h=-l}^l \sum_{k=-m}^m f_{hk} L_r^h L_c^k.$$

As an alternative specification to the SAR model, Besag (1974) introduced the conditional spatial autoregressive (CAR) model, whose expression, based on the first two conditional moments, is

$$E(y_{ij} | y_{h \neq i, k \neq j}) = \sum_{h=-l}^l \sum_{k=-m}^m a_{hk} y_{i+h, j+k}, \quad \text{VAR}(y_{ij} | y_{h \neq i, k \neq j}) = t^2$$

with $a_{00} = 0$ and where, to insure the symmetry of the process covariance matrix, the restriction $a_{hk} = a_{-h-k}$ for every h, k , has to be imposed.

Letting $e_{ij} = y_{ij} - E(y_{ij} | y_{h \neq i, k \neq j})$, the CAR model can be given expression

$$a(\mathbf{L})y_{ij} = e_{ij}, \quad \text{where } a(\mathbf{L}) = 1 - \sum_{h=-l}^l \sum_{k=-m}^m a_{hk} L_r^h L_c^k.$$

As is well known, (see, e.g., Cliff and Ord, 1981) for non causal models the conditional and simultaneous specifications with equal lag polynomial and innovation variance are not equivalent, as is the case for causal models. It is, however, possible to establish an equivalence relationship between SAR models and higher order CAR models. More specifically, letting $p = |l| + |m|$ denote the order of a given SAR model, Whittle (1986) proves that to each $\text{SAR}(p)$ model corresponds a CAR model of order $2p$, whose parameters are non linear functions of the $\text{SAR}(p)$ model parameters, obtained by equating the corresponding potential matrices.

On the infinite planar lattice this amounts to impose that the lag polynomial of the CAR model satisfies the following polynomial equation

$$(1) \quad a(\mathbf{L}) = [f(\mathbf{L})]^2$$

with normalizing constraint $a_{hk} = 1$ if $h=k=0$.

Equating terms of the same order in both sides of (1), a system of non linear equations is obtained

$$a_{hk} = g_{hk}(f), \quad h = -l(p), \dots, l(p), k = -m(p), \dots, m(p)$$

defining the constraints that the coefficients of the $\text{CAR}(2p)$ specification must satisfy for it to be equivalent to a given $\text{SAR}(p)$ specification. The conditional variance is finally identified by the normalization constraint $a_{00} = 1$.

As an illustrative example, in the case of the isotropic $\text{SAR}(1)$ model we have

$$f(\mathbf{L}) = 1 - f(L_r + L_r^{-1} + L_c + L_c^{-1}),$$

$$[f(\mathbf{L})]^2 = 1 + 4f^2 - 2f(L_r + L_r^{-1} + L_c + L_c^{-1}) +$$

$$+ 2f^2(L_r L_c + L_r^{-1} L_c + L_r L_c^{-1} + L_r^{-1} L_c^{-1}) + f^2(L_r^2 + L_r^{-2} + L_c^2 + L_c^{-2})$$

where, upon division by $(1 + 4f^2)$, the following three equations system obtains, stating the constraints that the equivalent CAR(2) coefficients must satisfy

$$\begin{cases} a_{10} = a_{01} = \frac{2f}{1 + 4f^2} \\ a_{11} = a_{1-1} = -\frac{2f^2}{1 + 4f^2} \\ a_{20} = a_{02} = -\frac{f^2}{1 + 4f^2} \end{cases}$$

and where conditional variance is $t^2 = (1 + 4f^2)^{-1} s^2$.

Based on the above stated equivalence, the SAR(p) model with coefficients f can be written in the equivalent CAR(2 p) form with expression

$$E(y_{ij} | y_{h \neq i, k \neq j}) = \sum_{h=-2l}^{2l} \sum_{k=-2m}^{2m} g_{hk}(f) y_{i+h, j+k} = \mathbf{g}'(f) \mathbf{z}_{ij}$$

$$VAR(y_{ij} | y_{h \neq i, k \neq j}) = f(f, s^2)$$

For a sample of observations collected over an $R \times C$ grid, the least squares estimator of the SAR coefficients can be derived by minimizing the objective function

$$SS(f) = \sum_{i=1}^R \sum_{j=1}^C [y_{ij} - \mathbf{g}'(f) \mathbf{z}_{ij}]^2$$

Given the non linearity of the functions $g_{hk}(\cdot)$, the minimization of the residual sum of squares can be achieved by means of the non linear least squares (NLLS) estimator, that can be computed iteratively on the basis of a first order Taylor expansion of the regression function around a given value for the parameter vector f_0 (see, e.g., Greene, 2000). In this case the iterative solution to the minimization problem is given by

$$\hat{f}_{t+1} = \hat{f}_t + \left[\sum_{i=1}^R \sum_{j=1}^C \mathbf{x}_{ij}^t \mathbf{x}_{ij}^{t'} \right]^{-1} \left[\sum_{i=1}^R \sum_{j=1}^C \mathbf{x}_{ij}^{t'} (y_{ij} - \mathbf{g}'(\hat{f}_t) \mathbf{z}_{ij}) \right]$$

where the pseudo-regressors $\mathbf{x}_{ij}^t$ are defined to be $\mathbf{x}_{ij}^t = \frac{\partial \mathbf{g}'(f) \mathbf{z}_{ij}}{\partial f} \Big|_{f=\hat{f}_t}$.

The NLLS estimator is consistent if the following condition holds (Greene, 2000)

$$p \lim \frac{1}{RC} \sum_{i=1}^R \sum_{j=1}^C \mathbf{x}_{ij}^0 e_{ij} = 0$$

where $\mathbf{x}_{ij}^0$ is the vector of pseudo-regressors evaluated at the true parameter vector $\mathbf{f}_0$ and $\mathbf{e}_{ij} = y_{ij} - \mathbf{g}'(\mathbf{f}_0)\mathbf{z}_{ij}$. The condition imposes the asymptotic orthogonality of the pseudo-regressors to the error term, and is satisfied in this case, where $\mathbf{x}_{ij}^0$ is linear in $\mathbf{z}_{ij}$, given the fact that in the CAR model $\mathbf{z}_{ij} \perp \mathbf{e}_{ij}$ by definition.

3. A Monte Carlo experiment

The performance of the NLLS estimator in finite samples has been evaluated through a series of Monte Carlo experiments. To allow for comparisons with the performance of the ML estimator, the analysis has been conducted with reference to the isotropic SAR(1) specification, for which, using the analytical expression for the eigenvalues of the spatial contiguity matrix provided by Ord (1975), the ML estimator is computationally feasible also for large lattices.

Experiments have been carried out allowing for different value of the autoregressive coefficient and for increasing sample size and performing 1.000 replications for each cell. The mean estimates are displayed in Table 1 with standard errors in parentheses.

Table 1: *The Monte Carlo simulation results: estimated bias (standard error in parentheses)*

Coefficient value	Lattice size							
	10×10		25×25		50×50		100×100	
	NLLS	ML	NLLS	ML	NLLS	ML	NLLS	ML
$f=0.1$	-0.0067 (0.0675)	0.0075 (0.0345)	-0.0003 (0.0144)	0.0010 (0.0127)	-0.0002 (0.0070)	0.0003 (0.0063)	-0.0000 (0.0034)	0.0001 (0.0030)
$f=0.2$	-0.0180 (0.0695)	0.0109 (0.0233)	-0.0027 (0.0155)	0.0017 (0.0076)	-0.0008 (0.0069)	0.0004 (0.0037)	-0.0003 (0.0034)	0.0001 (0.0018)
$f=0.225$	-0.0252 (0.0612)	0.0108 (0.0188)	-0.0043 (0.0170)	0.0016 (0.0053)	-0.0016 (0.0071)	0.0004 (0.0025)	-0.0005 (0.0033)	0.0001 (0.0012)
$f=0.245$	-0.0180 (0.0668)	0.0109 (0.0182)	-0.0027 (0.0170)	0.0017 (0.0054)	-0.0008 (0.0073)	0.0004 (0.0026)	-0.0003 (0.0034)	0.0001 (0.0012)

For both estimators the bias tends to zero as the sample size increases. While for large samples and smaller values of the autoregressive coefficient the NLLS estimator is almost as efficient as the ML estimator, in smaller samples and for values of the f coefficient close to the boundary of the admissibility region the performance of the NLLS estimator is inferior to maximum likelihood in relative terms, although still remarkable in absolute terms. Overall the NLLS estimator seems to provide, analogously to the pseudo-likelihood estimator in the case of the CAR model, a valuable alternative to the ML approach, useful when the ML estimator is not computationally feasible as, for example in the case of higher order or anisotropic specifications.

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