

A LM specification test for GARCH-BL models ¹

Un test di specificazione LM per modelli GARCH-BL

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Riassunto: In questo lavoro viene presentato un nuovo test LM ad alternativa specificata per la verifica dell'ipotesi di effetti asimmetrici nella varianza condizionata a fronte di un'ipotesi nulla di tipo GARCH. Il modello alternativo considerato è dato dal GARCH-BL recentemente proposto da Storti and Vitale (2001). Le proprietà della statistica test proposta vengono analizzate attraverso uno studio di simulazione.

Keywords: leverage effect, financial time series, Lagrange Multiplier test, likelihood.

1. Introduction

The GARCH Bilinear (GARCH-BL) model, recently proposed by Storti and Vitale (2001), allows to capture asymmetric patterns, such as leverage effects, in the volatility of financial time series by augmenting the standard GARCH model with interactions between past shocks and volatilities. Although the statistical properties of this class of models have been widely discussed by Storti and Vitale (2001), a formal testing procedure for verifying the hypothesis of asymmetry induced by a specific GARCH-BL model is still lacking. To this purpose, in this paper a new LM test statistic is proposed and its small sample properties are analysed by means of a simulation study. The structure of the paper is as follows. Section 2 presents the GARCH-BL model while the testing procedure is discussed in section 3. Section 4 concludes.

2. The GARCH-Bilinear model

Let u_t be an observed time series such that $u_t \sim (0, \sigma_u^2)$. A GARCH-BL model for u_t is specified by the following equations

$$\begin{aligned} u_t &= h_t z_t \\ h_t^2 &= a_0 + \sum_{i=1}^r a_i u_{t-i}^2 + \sum_{j=1}^s b_j h_{t-j}^2 + \sum_{k=1}^{r^*} c_k u_{t-k} h_{t-k} \end{aligned} \quad (1)$$

where $r^* = \min\{r, s\}$ and z_t , $t = 1, \dots, T$, is a series of iid random variables with zero mean and unit variance. It follows that the observation process u_t is serially uncorrelated with $(u_t | I^{t-1}) \sim (0, h_t^2)$, where I^{t-1} denotes past information up to time $(t-1)$. In model (1) leverage effects are explained by the interactions between past observations

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and volatilities. More precisely, for $c_k < 0$, a positive quantity is added to h_t^2 if $u_{t-k} < 0$ while the same quantity is subtracted if $u_{t-k} > 0$. It is interesting to note how the magnitude of the effect will depend not only on the value of u_{t-k} but also on the value of h_{t-k} . The GARCH-BL model can be considered as a generalization of the conventional GARCH model which can be obtained as a special case of the GARCH-BL. The denomination *Bilinear* is motivated by the interaction terms included into the conditional variance equation (1). Under the assumption of normality of the standardized innovations z_t , the log-likelihood function of the general GARCH-BL model can be written as:

$$\ell(u; \boldsymbol{\psi}) = -\frac{T}{2} \log 2\pi - \frac{1}{2} \sum_{t=1}^T \log h_t^2 - \frac{1}{2} \sum_{t=1}^T \frac{u_t^2}{h_t^2} \quad (2)$$

where $\boldsymbol{\psi}$ is a vector containing all the unknown parameters in (1). The log-likelihood function (2) can be maximized by means of an EM type algorithm. The main reason for using the EM algorithm is that it allows to obtain parameter estimates which naturally guarantee the positive definiteness of the conditional variance with no need for additional parameter constraints. An outline of the estimation strategy is given in Storti (1999).

3. A Lagrange Multiplier test statistic

In this section we derive the LM test statistic for verifying the null hypothesis of a GARCH(r,s) against the alternative of a GARCH-BL model of the same order. The Lagrange Multiplier test statistic can be written in the usual form

$$LM = \left[s(\tilde{\boldsymbol{\psi}}_0) \right]' I(\tilde{\boldsymbol{\psi}}_0)^{-1} \left[s(\tilde{\boldsymbol{\psi}}_0) \right]$$

where $s(\boldsymbol{\psi}) = \frac{\partial \ell}{\partial \boldsymbol{\psi}}$ is the score function and $I(\boldsymbol{\psi}) = -E \left[\frac{\partial^2 \ell}{\partial \boldsymbol{\psi} \partial \boldsymbol{\psi}'} \right]$ the information matrix, both to be evaluated at the restricted estimate $\tilde{\boldsymbol{\psi}}_0$. Under the assumption of conditional normality, assuming that h_1^2 is fixed so that $\partial h_1^2 / \partial \boldsymbol{\psi} = \mathbf{0}$, it can be shown that:

$$s(\tilde{\boldsymbol{\psi}}_0) = \sum_{t=1}^T \frac{1}{2h_{0t}^2} \left[\frac{u_t^2}{h_{0t}^2} - 1 \right] \left[\frac{\partial h_t^2}{\partial \boldsymbol{\psi}} \right]_{H_0}$$

where h_{0t}^2 is the estimated conditional variance under the null hypothesis of GARCH and $\left[\frac{\partial h_t^2}{\partial \boldsymbol{\psi}} \right]_{H_0}$ is the first derivative of the conditional variance of the alternative model evaluated under the null. Also, the information matrix under the null is consistently estimated by

$$\tilde{I}(\tilde{\boldsymbol{\psi}}_0) = \frac{1}{2} \sum_{t=1}^T \left\{ \frac{1}{h_{0t}^2} \left[\frac{\partial h_t^2}{\partial \boldsymbol{\psi}} \right]_{H_0} \right\} \left\{ \frac{1}{h_{0t}^2} \left[\frac{\partial h_t^2}{\partial \boldsymbol{\psi}} \right]_{H_0} \right\}'.$$

The derivatives $\frac{\partial h_t^2}{\partial \boldsymbol{\psi}}$ can be evaluated recursively observing that the conditional variance h_t^2 can be expressed as a quadratic form in the following way

$$h_t^2 = \mathbf{C}_t \mathbf{Q} \mathbf{C}_t' + a_0 \quad (3)$$

where

$$\mathbf{C}_t \begin{cases} = [u_{t-1}, h_{t-1}, \dots, u_{t-r^*}, h_{t-r^*}, u_{t-(r^*+1)}, \dots, u_{t-r}] & \text{for } r > s \\ = [u_{t-1}, h_{t-1}, \dots, u_{t-r}, h_{t-r}] & \text{for } r = s \\ = [u_{t-1}, h_{t-1}, \dots, u_{t-r^*}, h_{t-r^*}, h_{t-(r^*+1)}, \dots, h_{t-s}] & \text{for } r < s \end{cases}$$

and $\mathbf{Q}$ has the following block diagonal structure:

$$\begin{pmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & \mathbf{S} \end{pmatrix}$$

with

$$\mathbf{R} = \begin{pmatrix} \mathbf{R}_1 & \dots & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \ddots & \mathbf{0} & \dots & \mathbf{0} \\ \vdots & \dots & \ddots & \dots & \vdots \\ \mathbf{0} & \dots & \dots & \dots & \mathbf{R}_{r^*} \end{pmatrix} \quad \mathbf{S} = \begin{pmatrix} s_1 & \dots & 0 & \dots & 0 \\ 0 & \ddots & 0 & \dots & 0 \\ \vdots & \dots & \ddots & \dots & \vdots \\ 0 & \dots & \dots & \dots & s_q \end{pmatrix}$$

where, for $i = 1, \dots, r^*$, $\mathbf{R}_i = \begin{pmatrix} a_i & \frac{1}{2}c_i \\ \frac{1}{2}c_i & b_i \end{pmatrix}$ and, for $i = 1, \dots, q$, with $q = \max\{r, s\} - r^*$, $s_i = a_{r^*+i}$, if $r > s$, or $s_i = b_{r^*+i}$, if $r < s$; if $r = s$, $\mathbf{Q} = \mathbf{R}$. Hence, we can write

$$\frac{\partial h_t^2}{\partial \psi_i} = \frac{\partial \mathbf{C}_t}{\partial \psi_i} \mathbf{Q} \mathbf{C}_t' + \mathbf{C}_t \frac{\partial \mathbf{Q}}{\partial \psi_i} \mathbf{C}_t' + \mathbf{C}_t \mathbf{Q} \frac{\partial \mathbf{C}_t'}{\partial \psi_i} + \frac{\partial a_0}{\partial \psi_i}$$

for $i = 1, 2, \dots, k$ ($k = r + s + r^* + 1$).

Under the null H_0 , if the usual regularity conditions hold, the LM test statistic has an asymptotic χ^2 distribution with r^* degrees of freedom. An asymptotically equivalent test statistic (see e.g. Engle and Ng, 1993) can be calculated as $LM_R = T \times R^2$ where R^2 is the squared multiple correlation coefficient from the regression of

$$\mathbf{y}_t = \left(\frac{u_t^2}{h_{0t}^2} - 1 \right) \quad \text{on} \quad \mathbf{x}_t = \frac{1}{h_{0t}^2} \left[\frac{\partial h_t^2}{\partial \psi'} \right]_{H_0} \quad (t = 1, \dots, T).$$

For the case of the GARCH-BL(1,1) model, the small sample properties of the LM_R test statistic have been assessed by means of a simulation study. More precisely the empirical size (table 1) and power (table 2) have been evaluated over 500 Monte Carlo runs for different sample sizes ($T=300, 500$), nominal significance levels (1%, 5%, 10%) and parameter combinations chosen to reproduce different levels of persistence and of asymmetry in the conditional variance dynamics. All the series have been simulated from conditionally Gaussian processes.

4. Concluding remarks

The results of the Monte Carlo simulation study here performed show that the proposed LM test for GARCH-BL type asymmetry has satisfactory small sample properties. The

Table 1: Simulated size for the LM_R test statistic over 500 Monte Carlo runs for $T=\{300,500\}$, 3 nominal significance levels (1%, 5% and 10%) and 3 different sets of parameter combinations (**H**, **M**, **L**).

Parameter Values	T	10%	5%	1%
H	300	10.8	5	1.12
$(a_0 = 0.01, a_1 = 0.09, b_1 = 0.90)$	500	13.4	6.8	1.4
M	300	10.6	6.4	1.6
$(a_0 = 0.01, a_1 = 0.10, b_1 = 0.80)$	500	11.8	5.4	0.6
L	300	12.4	8	2.8
$(a_0 = 0.01, a_1 = 0.10, b_1 = 0.70)$	500	10.4	5.2	0.6

Table 2: Simulated power for the LM_R test statistic over 500 Monte Carlo runs for $T=\{300,500\}$, 3 nominal significance levels (1%, 5% and 10%) and 3 different sets of parameter combinations (**HS**, **MS**, **LS**).

Parameter Values	T	10%	5%	1%
HS	300	24	14.2	6.2
$(a_0 = 0.01, a_1 = 0.09, b_1 = 0.90, c_1 = -0.03)$	500	28	17.8	7.6
MS	300	77.4	67.4	41.6
$(a_0 = 0.01, a_1 = 0.10, b_1 = 0.80, c_1 = -0.15)$	500	95	92	79.8
LS	300	94.6	90.4	74.4
$(a_0 = 0.01, a_1 = 0.10, b_1 = 0.70, c_1 = -0.25)$	500	98.4	98.2	95.6

empirical size of the test for both the sample sizes considered is always close to its nominal value. For what concerns the power of the test, the results can be also considered satisfactory and tend to improve as the sample size increases from $T=300$ to $T=500$. The worst results have been obtained for the model HS characterized by a high persistence and a low value of the asymmetry parameter ($c_1=-0.03$) while, as expected, the empirical power increases as the impact of asymmetry becomes more substantial. Finally, it is worth noting that the test statistic presented in the paper is based on the assumption of conditional normality. Hence it would be interesting to analyse the robustness properties of the testing procedure when the normality assumption does not hold.

References

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