

# A general framework for fitting stochastic volatility models

*Un approccio generale per la stima dei modelli a volatilità stocastica*

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**Riassunto:** In questo lavoro si propone un metodo per stimare i modelli a volatilità stocastica. Innanzitutto si mostra come, utilizzando il metodo della quadratura, la verosimiglianza di ognuno dei modelli che cade nella classe in questione può essere adeguatamente approssimata tramite una funzione calcolabile sulla base di una relazione ricorsiva. La verosimiglianza approssimata può essere facilmente derivata rispetto ai parametri del modello e quindi massimizzata tramite algoritmi iterativi standard, come l'algoritmo Newton-Raphson. Inoltre, possono essere stimati gli errori standard da associare alle stime e costruiti intervalli di confidenza per i parametri. L'accuratezza delle stime, degli standard error e degli intervalli di confidenza viene esaminata tramite uno studio basato su una serie di simulazioni.

**Keywords:** High-dimensional integral, Maximum likelihood estimation, Newton-Raphson algorithm, Quadrature.

## 1. Introduction

The econometrics of financial markets acknowledges as characteristic feature of financial time series a changing volatility. The literature on this topic has focused on two class of models: the ARCH-type models, which are certainly the most popular and are straightforward to estimate, and the *stochastic volatility* (SV) models that, though less popular, have the advantage to be easily interpreted. Within the latter models, in particular, volatility, which is an unobservable factor, is expressed through a latent process modelled together with the process of the returns. The simplest SV model, known as *basic SV model*, is based on the assumptions of (i) Gaussianity of the latent process and the process of the returns and (ii) uncorrelation between the two processes. Relaxing (i) or (ii) allows one to take into account, respectively, a possible skewness or leptokurtosis in the distribution of the returns or a non-symmetric response of the volatility to the shocks.

Maximum likelihood (ML) estimation of the parameters of a SV model is troublesome. So, alternative estimation methods have been proposed. The most popular are:

1. *Generalized Method of Moments*, initially applied by Melino and Turnbull (1990). Refinements are represented by the *Efficient Method of Moments* whose performance was analyzed by Andersen *et al.* (1999);
2. *Quasi-likelihood method*, proposed by Harvey and Shephard (1996) and based on a linear state-space model;
3. *Bayesian methods*, introduced by Jacquier *et al.* (1995).

The present paper proposes a general framework that, following a ML approach and exploiting the quadrature method of finite integrals, allows one to fit any SV model. This approach has been introduced by Fridman and Harris (1998) and extended by Bartolucci and De Luca (2001) who analytically computed the first and second derivatives of the approximated likelihood.

In the following Section we examine the class of SV models whereas in Sections 3 and 4 we describe our approach. We also performed a simulation study which makes use of time series with different sizes generated under several SV models. The results, not summarized here, show that our approach ensures reasonably accurate estimates and reliable standard errors and confidence intervals.

## 2. The class of SV models

The basic assumption of any SV model is that the changes of the log-prices, and hence of the returns, of a financial asset are affected by the flow of information arriving into the financial markets. A latent variable is defined in order to take such information into account; by assumption, this latent variable affects only the variance of the process of the returns. More precisely, a SV model can be expressed as

$$\begin{aligned} y_t &= \exp\left(\frac{1}{2}h_t\right) \varepsilon_t \\ h_t &= \gamma_0 + \gamma_1 h_{t-1} + \sigma_\eta \eta_t \end{aligned}$$

where  $y_t$  and  $h_t$  are, respectively, the value of the series and of the latent variable at time  $t$  ( $t = 1, \dots, T$ ) while  $\gamma_0$ ,  $\gamma_1$  and  $\sigma_\eta$  are parameters to be estimated. Moreover, according to the assumptions on the distribution of  $\varepsilon_t$  and  $\eta_t$ , we have:

1. *Basic SV model:*  $\varepsilon_t$  and  $\eta_t$  have both normal distribution with null mean and are uncorrelated. This model, also known as *log-normal* SV model, is the simplest one and hence is widely used; it has been introduced by Taylor (1986). A possible refinement could be represented by the introduction of some covariates.
2. *Asymmetric SV model:*  $\varepsilon_t$  and  $\eta_t$  have joint normal distribution with null mean and variance-covariance matrix with diagonal elements equal to 1 and off-diagonal elements equal to  $\rho < 0$ . This model has been introduced by Hull and White (1987). Its denomination depends on the asymmetric response of  $h_t$  to positive and negative value of  $\varepsilon_t$  of the same magnitude, because of the effect on  $\eta_t$ . Note that this is a generalization of the basic SV model that holds when  $\rho = 0$ . Then a test for lack of asymmetry can be carried out through a  $t$ -test on the parameter  $\rho$ . Such a model has not found a widespread diffusion because of computational problems concerning its estimation.
3. *Generalized SV models:* a non-Gaussian distribution is considered for  $\varepsilon_t$  and/or  $\eta_t$ . In order to match the feature of fat-tails of financial returns, Fridman and Harris (1998) suggested to use the  $t$  distribution for  $\varepsilon_t$ , whereas, to take into account the possible skewness of the distribution of the returns, the skew-normal distribution was proposed by Bartolucci *et al.* (2000). In both cases, the basic SV model can be obtained as a particular case. The  $t$  distribution tends to a Normal one when the number of degrees of freedom tends to infinity whereas the Skew-normal distribution reduces to a Normal one when the skewness parameter is equal to zero.

### 3. The general framework

The likelihood of a SV model is obtained by integrating the latent variable out, as follows

$$L(\boldsymbol{\theta}) = \int_{\mathcal{R}^T} \left[ \prod_{t=1}^T f(y_t, h_t | h_{t-1}, \boldsymbol{\theta}) \right] d\mathbf{h}$$

where  $f(y_t, h_t | h_{t-1}, \boldsymbol{\theta})$  is the joint density of  $y_t$  and  $h_t$  given  $h_{t-1}$  and  $\boldsymbol{\theta}$ , whereas  $\mathbf{h} = (h_1 \cdots h_T)'$ . The above  $T$ -dimensional integral may not be computed exactly. However, on the basis of the quadrature method, we may approximate  $L(\boldsymbol{\theta})$  with the required precision through the function

$$\tilde{L}(\boldsymbol{\theta}) = a^T \sum_{i_1=1}^n \left[ \sum_{i_2=1}^n \cdots \left[ \sum_{i_T=1}^n f(y_T, x_{i_T} | x_{i_{T-1}}, \boldsymbol{\theta}) \right] \cdots f(y_2, x_{i_2} | x_{i_1}, \boldsymbol{\theta}) \right] f(y_1, x_{i_1} | h_0, \boldsymbol{\theta}),$$

where  $x_1, \dots, x_n$  is an appropriate set of quadrature points and  $a = (x_n - x_1)/(n - 1)$ . This approach represents an extension of the method proposed by Fridman and Harris (1998) and Bartolucci and De Luca (2001) which may be used only to deal with the basic SV model. Note that

$$\tilde{L}(\boldsymbol{\theta}) = a^T \mathbf{1}' \mathbf{F}_T \mathbf{F}_{T-1} \cdots \mathbf{F}_2 \mathbf{f}_1, \quad (1)$$

where  $\mathbf{f}_1$  is the vector with entries  $f(y_1, x_i | h_0)$ , for  $i = 1, \dots, n$ , and  $\mathbf{F}_t$  is the matrix with entries  $f(y_t, x_i | x_j)$ , for  $i, j = 1, \dots, n$ . The approximate likelihood (1) may be also expressed as  $\tilde{L}(\boldsymbol{\theta}) = a^T \mathbf{1}' \mathbf{f}_T$ , where  $\mathbf{f}_t$ , for  $t = 2, \dots, T$ , is defined through the recursion

$$\mathbf{f}_t = \mathbf{F}_t \mathbf{f}_{t-1}$$

that may be easily evaluated. Similarly, we may compute the first and second derivatives of  $\tilde{L}(\boldsymbol{\theta})$ . The parameter vector that maximizes  $\tilde{L}(\boldsymbol{\theta})$  is obtained after running the Newton-Raphson algorithm implemented on the basis of such derivatives. It is an approximated ML estimate and will be denoted by  $\hat{\boldsymbol{\theta}}$ .

This is how the vector and matrices involved in the computation of  $\tilde{L}(\boldsymbol{\theta})$  specialize for any of the models outlined in Section 2:

1. *Basic SV model*: in this case the factorization

$$f(y_t, h_t | h_{t-1}, \boldsymbol{\theta}) = f_Y(y_t | h_t) f_H(h_t | h_{t-1}, \boldsymbol{\theta}) \quad (2)$$

holds, where  $f_Y$  and  $f_H$  refer to the Gaussian distributions of the processes  $\{y_t\}$  and  $\{h_t\}$ , respectively. This implies that the matrix  $\mathbf{F}_t$  may be computed as  $\text{diag}(\mathbf{l}_t) \mathbf{F}$ , where  $\mathbf{l}_t$  is the vector with entries  $f_Y(y_t | x_i)$ , for  $i = 1, \dots, n$ , and  $\mathbf{F}$  is the matrix with entries  $f_H(x_i | x_j, \boldsymbol{\theta})$ , for  $i, j = 1, \dots, n$ . Therefore there is a strong simplification from a computational point of view; see Bartolucci and De Luca (2001) for details.

2. *Asymmetric SV model*: in this case the factorization (2) does not longer hold and so  $\mathbf{F}_t$  has to be computed at any step of the above recursion. Each entry,  $f(y_t, x_i | x_j)$ , is the conditional bivariate Gaussian density of  $y_t$  and  $x_i$ , given  $x_j$  and  $\boldsymbol{\theta}$ .
3. *Generalized SV models*: the steps followed for the basic SV model can be repeated. The difference is that the entries of  $\mathbf{l}_t$  and/or  $\mathbf{F}$  refer to non-Gaussian distributions.

#### 4. Inference on the parameters

Exploiting the proposed approach, it is possible to construct confidence intervals as well as testing hypotheses of interest. In fact, an estimator of the covariance matrix of  $\hat{\theta}$  may be obtained as

$$\widehat{\mathbf{V}} = \left( - \frac{\partial^2 \log \tilde{L}(\theta)}{\partial \theta^2} \Big|_{\hat{\theta}} \right)^{-1}$$

so that  $\widehat{\text{s.e.}}(\hat{\theta}_i) = \sqrt{\hat{v}_{ii}}$ , where  $\hat{v}_{ii}$  is the  $i$ -th element on the diagonal of  $\widehat{\mathbf{V}}$ , and the 95% confidence interval for  $\theta_i$  is equal to  $\hat{\theta}_i \pm 1.96 \widehat{\text{s.e.}}(\hat{\theta}_i)$ . An example of hypothesis that one usually wishes to test is that of lack of asymmetry of the log-returns when the asymmetric SV model is adopted. This hypothesis may be expressed as  $H_0 : \rho = 0$  and may be tested, against  $H_1 : \rho \neq 0$  or  $H_1 : \rho < 0$ , on the basis of the statistic  $t = \hat{\rho} / \widehat{\text{s.e.}}(\hat{\rho})$ .

#### References

- Andersen T., Chung H. and Sorensen B. (1999) Efficient method of moments estimation of a stochastic volatility model: a monte carlo study, *Journal of Econometrics*, 91, 61–87.
- Bartolucci F. and De Luca G. (2001) Maximum likelihood estimation of a latent variable time-series model, *Applied Stochastic Models in Business and Industry*, 17, 5–17.
- Bartolucci F., De Luca G. and Loperfido N. (2000) A generalization for skewness of the basic stochastic volatility model, *Proceedings of the 15th International Workshop on Statistical Modelling*, 140–145.
- Fridman M. and Harris L. (1998) A maximum likelihood approach for non-gaussian stochastic volatility models, *Journal of Business & Economic Statistics*, 16, 284–291.
- Harvey A. and Shephard N. (1996) Estimation of an asymmetric stochastic volatility model for asset returns, *Journal of Business & Economic Statistics*, 14, 429–434.
- Hull J. and White A. (1987) The pricing of options on assets with stochastic volatilities, *Journal of Finance*, 42, 281–300.
- Jacquier E., Polson N. and Rossi P. (1995) Bayesian analysis of stochastic volatility models, *Journal of Business & Economic Statistics*, 12, 371–417.
- Melino A. and Turnbull S. (1990) Pricing foreign currency options with stochastic volatility, *Journal of Econometrics*, 45, 239–265.
- Taylor S. (1986) *Modelling Stochastic Volatility*, John Wiley, Chichester.