

Testing unit roots in time series analysis through permutation methods¹

Le radici unitarie nell'analisi delle serie storiche attraverso i metodi di permutazione

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Riassunto: Nella teoria econometrica è di forte interesse indagare la presenza di radici unitarie nelle serie storiche in quanto è stata spesso evidenziata stazionarietà delle serie storiche economiche solo nelle differenze prime. In questo lavoro vogliamo proporre e studiare una tecnica per la verifica di ipotesi sulla presenza di radici unitarie da poter estendere anche applicando metodi di permutazione.

Keywords: unit roots, permutation tests.

1. Introduction

In econometric theory is very important to detect the presence of a unit root as the economic series often reveal stationarity only in the first differences. The tests, proposed in literature, give asymptotic unbiased critical regions. For example Dickey-Fuller test, see Dickey and Fuller (1979), and its transformations, or their *bootstrap* counterpart, see Procidano and Rigatti Luchini (2001), are very powerful under several alternatives. In this work we propose and study a technique usable in a permutation context. The use of permutation tests is often linked to two important features, that is the exchangeability of the observations and the comparisons of some groups. In order to develop this application, we propose some *ad-hoc* statistics dividing the observations in two groups. In the paper we remember the parametric properties of the test statistic and analyze some interesting features through simulation.

We consider the usual AR(1) model in which we assume Gaussian, homoschedastic and independent errors. The dynamic model can be explained by

$$y_t = \alpha y_{t-1} + \epsilon_t \quad (1)$$

where α is in $(-1, 1]$, $t = 0, 1, \dots, T$, $y_0 = 0$ and $E(\epsilon_t^2) = \sigma^2$ for each t .

The problem of detecting a unit root can be expressed by the following hypothesis system

$$\begin{cases} H_0 : & \alpha = 1 \\ H_1 : & \alpha < 1 \end{cases} \quad (2)$$

that is we want to test a nonstationary hypothesis against stationarity.

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2. The test procedure

On the basis of an observed trajectory $\{y_0, y_1, \dots, y_{T-1}, y_T\}$, $k = 1, 2, \dots, K$ and K be the integer part of $T/3$, let consider the following statistics

$$\xi_k^{(1)} = y_{3k} - y_{3k-1} \quad (3)$$

$$\xi_k^{(2)} = \frac{1}{\sqrt{2}}(y_{3k-1} - y_{3k-3}) \quad (4)$$

In this way, we form two groups, that we call “new” observations, each with K elements, whose distributions are identical under the null hypothesis. For example, when $\alpha = 1$, $y_{3k} = y_{3k-1} + \epsilon_{3k}$, so $y_{3k} - y_{3k-1} = \epsilon_{3k}$ has a normal distribution with zero mean and variance equal to σ^2 . In the same way, we can consider that $y_{3h-1} - y_{3h-2} + y_{3h-2} - y_{3h-3} = \epsilon_{3h-1} + \epsilon_{3h-2}$ is normally distributed with zero mean and variance equal to $2\sigma^2$, so that $\xi_h^{(2)}$, when $\alpha = 1$, has the same distribution of $\xi_k^{(1)}$, for each k and h .

Let note that the two groups of observations (3) and (4) consist of different values of ϵ_t for each k , so they are independent. In this case, as the means are obviously zero, we can work with variance estimators using $\zeta^{(1)} = 1/K \cdot \sum_{k=1}^K (\xi_k^{(1)})^2$ and $\zeta^{(2)} = 1/K \cdot \sum_{k=1}^K (\xi_k^{(2)})^2$ and their rate

$$F = \frac{\zeta^{(1)}}{\zeta^{(2)}}$$

Under H_0 in (2), the F statistic is distributed as an F distribution with (K, K) degrees of freedom. Apparently, the test statistic F tests the hypothesis system (2) rewritten in terms of variance in the following way

$$H_0 : \sigma_{\xi^{(1)}}^2 = \sigma_{\xi^{(2)}}^2 \quad (5)$$

When α is less than one, let observe that F assume values greater than one. In particular, it is easy to show that, for the first group, the variance of each “observation”, under an alternative of α , is

$$\text{var}(\xi_k^{(1)}) = \frac{2}{1+\alpha}\sigma^2$$

and, for the second group, the variance of $\xi_k^{(2)}$ is always equal to σ^2 . So, the rate between the variances is $\frac{2}{1+\alpha}$. It's interesting to note that for value of α close to one, even this fraction is close to one, so we think to make other statistics group, denoted by $\xi^{(l+1)}$, consisting of more than one residual term; in particular, denoting $y_h - y_{h-1}$ by e_h , their expression is

$$\xi_k^{(l+1)} = \frac{\sum_{j=1}^{l+1} e_{(l+2)*k-j}}{\sqrt{l+1}} \quad (6)$$

where k is an index with values in $\{1, \dots, K\}$ and $K = \left\lfloor \frac{T}{l+2} \right\rfloor$. In these cases too, under H_0 in (2), the $\xi_k^{(l+1)}$ are normally distributed, with zero mean, variance equal to σ^2 and maintain independence property. Furthermore, under a generic alternative $\alpha < 1$, we have

$$\text{var}(\xi_k^{(l+1)}) = \frac{2}{l+1}\sigma^2 \frac{1-\alpha^{l+1}}{1-\alpha^2} \quad (7)$$

In this case we can consider a statistic F_l equal to

$$F_l = \frac{\sum_{k=1}^K \left(\xi_k^{(1)} \right)^2}{\sum_{k=1}^K \left(\xi_k^{(l+1)} \right)^2} \quad (8)$$

with $K = \left\lfloor \frac{T}{l+2} \right\rfloor$, which is distributed again as F distribution when α is equal to 1. For each $l = 2, 3, \dots$, the distribution of (8), under an alternative hypothesis, is unknown and strongly depends on the dependence structure among the “observations”.

3. Some results and conclusions

The applied tests are studied by simulation, considering both the parametric and the permutation procedure, under different values of α and error distributions. The nominal level is fixed to 0.05 and the procedures are valued with increasing trajectories; in particular we fix T equal to 100, 150, 200, 300 and 500. The values of α are chosen in order to investigate close to the null hypothesis, as for other alternatives the empirical evidence does not need a test, in particular α takes values in $\{1, 0.98, 0.95, 0.9, 0.8, 0.7, 0.5, 0.2\}$.

- We apply the parametric test F on 1000 Monte Carlo simulations of the data, and we repeat the procedure until l equal to 8: indeed, as l increases, the degrees of freedom of statistic F vanish, with a consequent loss of power.
- To apply the permutation tests, assuming the exchangeability condition, let think to consider as “observations” the quantities $e_t = y_t - y_{t-1}$. We permute the first differences of the time series. As we know, permutation procedures depend strongly on sample sizes and in this case the number of observations is typically high, so we apply a randomization procedure. In fact, this method is known as *conditional Monte Carlo* (fundamentally different from *bootstrap*) and here we choose a number B of random permutations equal to 1000. The simulation results show a behaviour of the permutation test very close to the parametric one, according to theory that proves the convergence in probability of the permutation distribution of the statistic test to the true one, see Pesarin (2001).

As the parametric F test is very sensitive to the normality assumption, we perform some simulations with errors distributed as Student t with 4 degrees of freedom. The permutation test behaviour is considerably better than the parametric one.

Some interesting features regards the estimated significance level compared to the nominal one. We note a strong difference in the parametric test for small l , increasing with T , even for l equal to 8.

Table 1 shows the values of the relative differences $(\alpha_{obs} - 0.05)/0.05$: we can observe that the permutation test (see Table 1(b)) is closer to the nominal level than the parametric one (see Figure 1) in all cases. The number of Monte Carlo simulations is $n_{MC} = 1000$ in both cases.

The proposed procedure can be improved as regard the power function considering different test statistics, more sensitive to the subdivision of groups even for values of α in (1) very close to 1.

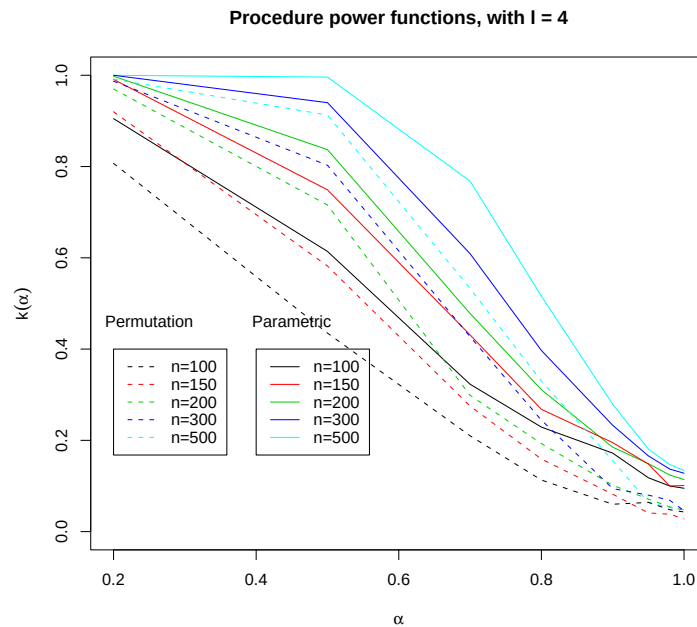
Table 1: *Relative difference between significance levels*

T	100	150	200	300	500
$l = 1$	1.38	1.80	2.32	2.22	2.38
$l = 2$	1.38	1.58	1.70	1.68	2.12
$l = 3$	1.38	0.90	1.34	1.26	1.96
$l = 4$	0.90	1.02	1.28	1.56	1.68
$l = 5$	0.84	1.10	1.20	1.26	1.34
$l = 6$	0.88	1.00	1.02	1.22	1.16
$l = 7$	0.66	0.42	0.72	1.10	1.48
$l = 8$	0.36	0.88	0.96	1.20	1.36

(a) *Parametric*

T	100	150	200	300	500
$l = 1$	-0.14	0.12	-0.08	-0.12	0.06
$l = 2$	0.18	-0.20	-0.14	0.14	0.20
$l = 3$	0.08	0.02	0.04	0.48	0.22
$l = 4$	-0.14	-0.44	-0.06	-0.08	-0.04
$l = 5$	-0.02	-0.04	0.04	0.20	0.02
$l = 6$	0.06	0.22	-0.12	0.20	0.02
$l = 7$	0.02	0.04	-0.02	0.16	0.06
$l = 8$	0.26	0.08	0.22	0.20	0.28

(b) *Permutation*

Figure 1: *Comparisons between parametric and permutation power functions, for $l = 4$.*

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