

Bootstrap Tests for Unit Roots in the Presence of Long-Memory Errors

Test Bootstrap per Radici Unitarie in Presenza di Errori a Memoria Lunga

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Riassunto: Nei lavori applicati il numero di volte che una serie temporale deve essere differenziata per raggiungere la stazionarietà viene usualmente determinato mediante metodi intuitivi e non formali. Tuttavia, per molte serie storiche osservate, prendere la differenza prima o seconda può risultare eccessivo. Un approccio alternativo è quello di utilizzare la classe dei più generali modelli frazionari ARFIMA(p, d, q), ma, in questo contesto, gli usuali test per radici unitarie hanno bassa potenza contro alternative frazionarie. In questo lavoro vedremo come utilizzare alcuni test asintotici di integrazione frazionaria per testare l'ipotesi nulla di radici unitarie contro alternative frazionarie, inoltre, poiché tali test risultano distorti in piccoli campioni, utilizzeremo una particolare metodologia *bootstrap* per migliorarne il comportamento.

Keywords: unit root, fractional ARIMA processes, bootstrap.

1. Introduction

In the empirical applications of Box-Jenkins ARIMA (p, d, q) models, the number of times that the observed time series must be differenced (parameter d) is usually determined by intuitive, but informal methods, for example the analysis of the empirical autocorrelation function of the differenced series. However, for many observed time series, taking the first or the second difference may be too strong. Another approach is to use models based on fractional integration, that permit the difference parameter d to assume non-integer values. Since the seminal paper of Granger and Joyeux (1980) and Hosking (1981), this type of model has received increasing attention because they provide a more flexible framework to study time series data. In particular, ARFIMA models can be used to model data dependence that is stronger than allowed in stationary ARMA process and weaker than implied by unit root processes. At the same time, they include the unit root hypothesis as a special case.

In a previous work Bisaglia and Procidano (2001) demonstrated that the Augmented Dickey-Fuller test is consistent against fractional alternatives, but it has low power. A bootstrap version of the Augmented Dickey-Fuller test, considered by the authors, helps to improve the power.

In this work we want to test the null hypothesis of unit root against fractional alternatives, considering a different class of tests usually employed in determining if a series has long-memory. These tests are asymptotic, so they often exhibit non negligible size distortion in small samples. To improve inference, in this work we suggest a bootstrap method to adjust critical values. We find out that bootstrap tests have better size properties than the corresponding asymptotic tests.

2. Testing for unit roots against fractional alternatives

A fractionally integrated autoregressive moving average (ARFIMA) model, $(x_t)_{t \in \mathbb{N}}$, is described by the equation $\Phi(B)(1-B)^d(X_t - m) = \Theta(B)e_t$, $t = 1, 2, \dots, T$, where all the roots of $F(B)$ and $Q(B)$ are outside the unit circle, $(e_t)_{t \in \mathbb{N}}$ are i.i.d. with zero mean and variance s^2 and m is the mean of the process. The differencing parameter d is the fractional parameter and it is allowed to assume any real number. If $d \in (-1/2, 1/2)$ the process is stationary and invertible, particularly, if $d \in (0, 1/2)$ the process is said to have long-memory, while if $d \in (-1/2, 0)$ the process is said antipersistent. If $d > 1/2$ the process becomes non stationary, but it is always possible to transform it into a stationary process by taking a suitable number of integer differences. Now, let $(x_t)_{t \in \mathbb{N}}$ be a real-valued time series whose behaviour follows the model:

$$x_t = x_{t-1} + h_t \text{ with } (1-B)^d h_t = e_t \quad (1)$$

where $d \in (-1/2, 1/2)$ and $(e_t)_{t \in \mathbb{N}}$ are white noise with zero mean and variance s^2 . Since it is possible to write (1) as $(1-B)x_t = h_t$, that is, x_t is a random walk with fractionally integrated innovations, it follows that $(1-B)^d x_t = e_t$ where $d = (1 + d) \in (1/2, 3/2)$. Then, any fractional process can be written as a process with an autoregressive unit root, but following the literature, only I(1) process, i.e. $d = 1$ (or equivalently $d = 0$), are said to contain a unit root. Intuitively, when $d \in (1/2, 1)$ the process x_t is non-stationary but less than a unit root process, while when $d \in (1, 3/2)$ the process has a somewhat greater degree of non-stationarity than a unit root process. Our problem is to test the null hypothesis $H_0: d = 1$ against fractional alternatives. To do that, we consider some tests for fractional integration that are applicable on stationary and invertible time series and we take the first differences of x_t . If x_t is a unit root process ($d = 1$) its first differences will be a white noise process and $d = 0$. If x_t is a fractionally integrated process then $d \in (1/2, 3/2)$ and we obtain a stationary fractional process, h_t , with fractional parameter $d \in (-1/2, 1/2)$. Thus the null hypothesis of unit root became $H_0: d=0$ in the first differences. The tests we consider are the rescaled range statistics, RS, derived by Hurst (1951), the modified rescaled range statistic, MRS derived by Lo (1991), the log-periodogram regression of Geweke and Porter-Hudack (1993), GPH, and a series of Lagrange multiplier type test developed by Robinson (1991) and Lobato and Robinson (1998), LM1, LM2 and LM3. Firstly, we want to see if these tests work better than the Dickey-Fuller and Augmented Dickey-Fuller tests under fractional alternatives. Secondly, since for some of these tests Cheung (1993) reports evidence of serious size distortions in small sample, our idea is to correct the distortions using bootstrap methods.

3. Bootstrap unit-root tests

The consistency of the bootstrap estimator of the distribution of the least square estimator of the autoregressive parameter in a simple first order autoregressive process, which may or may not have a stationary solution, has been investigated by several authors recently, nevertheless, it is not currently known if the bootstrap can improve the accuracy when the DGP has unit roots and long-memory. In this case, bootstrap methods based on sieve or other non parametric approximations of the data generating process, may work better. Our idea is to combine the wild bootstrap, that works well

when the DGP has a unit root (Procidano and Rigatti Luchini, 2001), with the sieve bootstrap (Bühlmann, 1997) that should work well when the DGP exhibits long-memory behaviour. The bootstrap scheme we followed is defined by the following steps:

1. Having observed the sample $x_1, x_2, \dots, x_T$, we take the first differences $h_t = \Delta x_t = x_t - x_{t-1}$.
2. We fit an autoregressive process to the first differences

$$\Delta x_t = b_1 \Delta x_{t-1} + b_2 \Delta x_{t-2} + \dots + b_{p(T)} \Delta x_{t-p(T)}$$

with $p(T) = o(T^{1/2+d})$. In practice we fixed $p_{max} = 10 \log_{10}(T)$ and then, following a suggestion of Bühlmann (1997), we chose the optimal p using the AIC criterion, $p = p_{AIC}$. Then, we obtain the estimated coefficients $\hat{b}_{1,T}, \hat{b}_{2,T}, \dots, \hat{b}_{p,T}$ using, for example, the Yule-Walker estimate.

3. Using $\hat{b}_{1,T}, \hat{b}_{2,T}, \dots, \hat{b}_{p,T}$ we construct a set of residuals like

$$e_{t,T} = \Delta x_t - \hat{b}_{1,T} \Delta x_{t-1} - \hat{b}_{2,T} \Delta x_{t-2} - \dots - \hat{b}_{p,T} \Delta x_{t-p}, \quad t = p+1, 2, \dots, T.$$

4. We extract elements of a random variable, e^* , with reintroduction $(T - p - 1)$ times, so that $E(e^*) = 0, E(e^{*2}) = 1$; values $e_t^*, t = p+1, 2, \dots, T$, are obtained.

5. Then we generate an *i.i.d.* sample $e_{t,T}^*, t = p+1, \dots, T$ using the $e_{t,T}$ and the e_t^* in the following way

$$e_{t,T}^* = \frac{|e_{t,T}| \cdot e_t^*}{\sqrt{1 - \text{diag}(P)}}$$

where $\mathbf{P} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$ and $\mathbf{X} = [x_{t-1}, \Delta x_{t-1}, \Delta x_{t-2}, \dots, \Delta x_{t-p}]$.

6. We generate bootstrap replicates $x_{t,T}^*$ using the $e_{t,T}^*$ according to

$$\Delta x_{t,T}^* = \hat{b}_{1,T} \Delta x_{t-1,T}^* + \hat{b}_{2,T} \Delta x_{t-2,T}^* + \dots + \hat{b}_{p,T} \Delta x_{t-p,T}^* + e_{t,T}^*, \quad t = p+1, 2, \dots, T$$

$$x_{t,T}^* = \sum_{j=1}^t \Delta x_{j,T}^*$$

7. We obtain the bootstrap version of the tests using the bootstrapped series.
8. Steps 4 – 7 are repeated a large number of times, yielding an empirical distribution for the bootstrap tests.

4. Results

In this section, we use some Monte Carlo experiments to investigate the size and the power of the described tests in finite samples. The Monte Carlo experiments were conducted for the following model $(1 - B)^d x_t = e_t$, $t = 1, 2, \dots, T$, with $d = 0.55, 0.7, 0.9, 1.0, 1.1, 1.3, 1.45$ and $e_t \sim i.i.d. N(0,1)$. Our results are based on 500 independent replications with $T = 100$ and 250 observations and for each simulated series we considered 500 bootstrap replications. The stationary fractional noise are generated using the recursive Durbin-Levinson algorithm and the non-stationary processes are generated autoregressively, $x_t = x_{t-1} + h_t$. All tests are two-sided with level 0.05. Table 1 presents the empirical power of the tests at a nominal 5% level of significance. The simulation results suggest that in general, the bootstrap testing procedure is able to improve the tests, in fact: i) every bootstrap test has better size properties than any of the original tests; ii) the bootstrap procedure does not impose distortions where the original tests are well-sized; iii) the power properties suggest that the bootstrap LM type tests are superior when testing for unit root against fractional integration in small samples; in larger samples all the LM type tests became equivalent. We conclude that

the bootstrap testing procedure provides a practical and effective method to improve the tests performances when testing for a unit root against fractional alternatives.

Table 1. Rejection percentage of the nominal 5% fractional integration test when the data follow an ARFIMA(0, δ ,0) process.

	RS	RS*	MRS	MRS*	GPH	GPH*	LM1	LM1*	LM2	LM2*	LM3	LM3*
T=100												
$d=0.55$	98.6	91.4	29.0	13.2	23.0	19.8	98.4	99.0	99.0	99.4	89.8	98.0
$d=0.7$	80.2	53.4	24.2	11.8	14.2	11.4	75.4	81.6	87.5	85.8	54.0	80.2
$d=0.9$	23.8	8.0	11.6	4.6	6.8	5.0	10.2	16.6	22.2	18.8	4.4	16.4
$d=1.0$	9.6	5.6	7.3	6.0	4.8	4.6	3.0	5.3	5.7	5.5	2.7	5.0
$d=1.1$	8.8	15.8	4.0	11.0	6.8	7.8	14.0	23.2	18.8	23.2	19.0	22.0
$d=1.3$	52.0	64.7	4.2	14.6	15.2	19.6	82.8	90.4	86.4	91.2	88.2	90.6
$d=1.45$	83.4	86.2	2.4	7.0	27.6	29.6	98.2	98.8	98.8	98.4	98.8	98.6
T=250												
$d=0.55$	100	100	67.2	47.8	41.4	38.0	100	100	100	100	100	100
$d=0.7$	96.8	91.8	48.6	31.4	25.8	21.0	99.8	99.8	100.0	99.9	99.4	99.8
$d=0.9$	28.0	15.8	16.6	9.6	7.8	6.6	37.2	41.0	51.6	42.6	26.2	39.8
$d=1.0$	7.7	5.3	6.5	5.4	5.8	6.0	4.3	6.3	7.6	6.9	4.5	6.0
$d=1.1$	21.6	27.0	11.4	17.6	7.6	9.2	43.6	51.4	49.8	51.8	50.4	52.6
$d=1.3$	79.2	84.4	23.2	33.2	26.6	32.2	99.6	99.8	99.8	99.8	99.8	99.8
$d=1.45$	97.0	97.8	12.0	18.8	55.4	56.4	100	100	100	100	100	100

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