

A “New” Two Term Subtractive Decomposition Of Herfindahl Concentration Measure

*Una “nuova” scomposizione sottrattiva in due termini
della misura di concentrazione di Herfindahl*

Paolo Radaelli, Michele Zenga

Dipartimento di Metodi Quantitativi per l’Economia
Facoltà di Economia - Università degli Studi di Milano-Bicocca
paolo.radaelli@unimib.it - michele.zenga@unimib.it

Riassunto: Nel presente lavoro si propone una scomposizione sottrattiva in due termini dell’indice di concentrazione di Herfindahl (H) valutato per le s aziende che operano in un settore industriale e che vendono i loro prodotti in k “regioni”. La scomposizione mostra come l’indice H calcolato sulle quote complessive di mercato delle aziende, può essere ottenuto sottraendo un termine positivo dalla media aritmetica ponderata degli indici H calcolati per le singole regioni. Tale termine può essere interpretato come una misura della dipendenza del fatturato delle aziende rispetto alle regioni. La stessa scomposizione può essere applicata anche all’indice di omogeneità di Gini nel caso di un collettivo composto da n individui classificati rispetto a due mutabili.

Keywords: Industrial Concentration, Herfindahl Index, Subtractive Decomposition, Homogeneity.

1. Introduction

In this paper we will analyze some properties of the Herfindahl¹ measure of concentration (H) in the case of s firms, of an industry I , selling their products in k regions. In particular, we will show that the index computed on the “overall” shares of the firms, can be obtained by subtracting a positive term from a weighted arithmetic mean of the indexes evaluated for each region. This term provides a measure of dependence of firms’ sales on the regions.

The existing literature focused mainly on the decomposability of inequality measures (Civardi 1987), (Schmid 1994), (Maasoumi 1999) and (Zenga 1986); some authors also proposed decompositions of H : (Clarke and Davies 1983), (Dagum 1997), (Piesch 1975) and (Zenga 2001). The decomposition we propose has the advantage that its components can be easily interpreted.

2. Notation

The i^{th} firm of the industry I is denoted with A_i , $i = 1, \dots, s$. The j^{th} region is denoted with R_j , $j = 1, \dots, k$. The amount of sales of A_i in R_j is denoted with x_{ij} .

¹For an introduction to the Herfindahl index the reader could refer to (Hall and Tideman 1967) and (Michellini and Pickford 1985).

The total sales of A_i are denoted with $T_{i\cdot}$ and are given by:

$$T_{i\cdot} = \sum_{j=1}^k x_{ij} \quad (1)$$

The total sales in the j^{th} region R_j are denoted with $T_{\cdot j}$ and are given by:

$$T_{\cdot j} = \sum_{i=1}^s x_{ij} \quad (2)$$

The total sales of the s firms in the k regions are denoted with $T_{\cdot\cdot}$ and are given by:

$$T_{\cdot\cdot} = \sum_{i=1}^s T_{i\cdot} = \sum_{j=1}^k T_{\cdot j} = \sum_{i=1}^s \sum_{j=1}^k x_{ij} \quad (3)$$

The Herfindahl index computed on the “overall” shares of firms is:

$$H(I) = \sum_{i=1}^s \left(\frac{T_{i\cdot}}{T_{\cdot\cdot}} \right)^2 \quad (4)$$

The Herfindahl index computed on the sales in R_j is:

$$H(R_j) = \sum_{i=1}^s \left(\frac{x_{ij}}{T_{\cdot j}} \right)^2 \quad (5)$$

$\overline{H}(R_j)$ denotes the weighted arithmetic mean of the indexes $H(R_j)$ with weights $\frac{T_{\cdot j}}{T_{\cdot\cdot}}$ (j^{th} region sales relative to the total sales of the industry I).

$$\overline{H}(R_j) = \sum_{j=1}^k H(R_j) \frac{T_{\cdot j}}{T_{\cdot\cdot}} \quad (6)$$

The amount of sales of A_i in R_j under the hypothesis of independence of A_i from R_j is $\hat{x}_{ij} = \frac{T_{i\cdot} T_{\cdot j}}{T_{\cdot\cdot}}$.

A measure of the dependence of firms' sales on the regions is:

$$D_{A/R} = \frac{1}{T_{\cdot\cdot}} \sum_{j=1}^k \frac{1}{T_{\cdot j}} \sum_{i=1}^s (x_{ij} - \hat{x}_{ij})^2 \quad (7)$$

The next sections show that subtracting $D_{A/R}$ from $\overline{H}(R_j)$ we obtain $H(I)$.

3. Marginal and conditional shares

The “total” share - marginal share - of A_i in the whole industry is given by the ratio:

$$q_{i\cdot} = T_{i\cdot} / T_{\cdot\cdot} \quad (8)$$

The “conditional” share of A_i in R_j is given by the ratio:

$$q_{i/j} = x_{ij} / T_{.j} \quad (9)$$

The “total” share - marginal share - of R_j in the whole industry is given by the ratio:

$$q_{.j} = T_{.j} / T_{..} \quad (10)$$

The share (8) is equal to the weighted arithmetic mean of the conditional shares $q_{i/j}$ with weights $q_{.j}$:

$$q_{i.} = \frac{1}{T_{..}} \sum_{j=1}^k x_{ij} = \frac{1}{T_{..}} \sum_{j=1}^k q_{i/j} T_{.j} = \sum_{j=1}^k q_{i/j} q_{.j} \quad (11)$$

4. Variance of the conditional shares and subtractive decomposition

A measure of the “dependence” of the sales of firm A_i on the regions may be obtained from the variance $V(q_{i/j})$ of the conditional shares $q_{i/j}$. From the results of the previous section we can write:

$$V(q_{i/j}) = \sum_{j=1}^k (q_{i/j} - q_{i.})^2 q_{.j} = \sum_{j=1}^k (q_{i/j})^2 q_{.j} - (q_{i.})^2 \quad (12)$$

Equality (12) holds for each firm A_i ($i = 1, \dots, s$); consequently we have:

$$\sum_{i=1}^s V(q_{i/j}) = \sum_{i=1}^s \sum_{j=1}^k (q_{i/j})^2 q_{.j} - \sum_{i=1}^s (q_{i.})^2 \quad (13)$$

From (8), (9), (10), (12) and (13) we obtain:

$$\sum_{i=1}^s \sum_{j=1}^k \left(\frac{x_{ij}}{T_{.j}} - \frac{T_{i.}}{T_{..}} \right)^2 \frac{T_{.j}}{T_{..}} = \sum_{i=1}^s \sum_{j=1}^k \left(\frac{x_{ij}}{T_{.j}} \right)^2 \frac{T_{.j}}{T_{..}} - \sum_{i=1}^s \left(\frac{T_{i.}}{T_{..}} \right)^2 \quad (14)$$

and, inverting the order of the summations:

$$\sum_{j=1}^k \frac{T_{.j}}{T_{..}} \sum_{i=1}^s \left(\frac{1}{T_{.j}} \right)^2 \left(x_{ij} - \frac{T_{i.} T_{.j}}{T_{..}} \right)^2 = \sum_{j=1}^k \frac{T_{.j}}{T_{..}} \sum_{i=1}^s \left(\frac{x_{ij}}{T_{.j}} \right)^2 - \sum_{i=1}^s \left(\frac{T_{i.}}{T_{..}} \right)^2 \quad (15)$$

From (4), (5) and (7) we get:

$$\frac{1}{T_{..}} \sum_{j=1}^k \sum_{i=1}^s \frac{1}{T_{.j}} (x_{ij} - \hat{x}_{ij})^2 = \sum_{j=1}^k H(R_j) \frac{T_{.j}}{T_{..}} - H(I) \quad (16)$$

where $\frac{1}{T_{..}} \sum_{j=1}^k \sum_{i=1}^s \frac{1}{T_{.j}} (x_{ij} - \hat{x}_{ij})^2 = \sum_{i=1}^s V(q_{i/j})$ is the “measure of dependence” of firms’ sales on the regions that we have denoted, in (7), with $D_{A/R}$.

Now from (6), (7) and (16) we get:

$$D_{A/R} = \overline{H}(R_j) - H(I) \geq 0 \quad (17)$$

An important result from (17) is that $\overline{H}(R_j) \geq H(I)$ with equality holding only if $x_{ij} = \hat{x}_{ij} : i = 1, \dots, s; j = 1, \dots, k$.

We conclude this section writing the “two term subtractive decomposition” for $H(I)$:

$$H(I) = \overline{H}(R_j) - D_{A/R} \quad (18)$$

References

- Civardi, M. (1987). *Proposte di scomposizione dell'indice del Gini - in La distribuzione personale del reddito: problemi di formazione, di ripartizione e di misurazione (a cura di Zenga M.) pp.26-47*, Vita e Pensiero.
- Clarke, R. and Davies, S. (1983). Aggregate concentration, market concentration and diversification, *The Economic Journal* **93**, 182–192.
- Dagum, C. (1997). Decomposition and interpretation of gini and the generalized entropy inequality measures, *Statistica* **LVII**, **3**, 295–308.
- Hall, M. and Tideman, N. (1967). Measures of concentration, *Journal of the American Statistical Association* **62**, 162–168.
- Leti, G. (2000). *Statistica Descrittiva*, il Mulino, Bologna.
- Maasoumi, E. (1999). *Multidimensional Approaches to welfare analysis - in Handbook of Income Inequality Measurement (edited by Silber J.) pp.437-484*, Kluwer Academic Publishers.
- Michellini, C. and Pickford, M. (1985). Estimating the herfindahl index from concentration ratio data, *Journal of the American Statistical Association* **80**, 301–305.
- Piesch, W. (1975). *Statistische Konzentrationsmasse: formale Eigenschaften u. verteilungstheoret. Zusammenhänge*, J.C.B. Mohr (Paul Siebeck) Tübingen.
- Schmid, F. (1994). *Decomposition of Inequality Measures - in Model and Measurement of Welfare and Inequality (edited by Eichhorn W.) pp.176-192*, Springer-Verlag.
- Zanella, A. (1988). *Lezioni di Statistica (Parte Seconda), Struttura dei dati in due o più dimensioni, La connessione*, Vita e Pensiero.
- Zenga, M. (1986). On the normalization, the invariance to the population replication and the additive decomposition in the $\{I^{(\alpha)}\}$ class of inequality measures, *Metron* **44**, 131–151.
- Zenga, M. (2001). A multiplicative decomposition of herfindahl concentration measure, *Metron* **59**, 3–10.