

# A test of Exponentiality based on the Mean Residual Life characterization

*Un test di Esponenzialità basato sulla caratterizzazione della Vita Media Residua*

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**Riassunto:** La vita media residua  $E(X - t|X > t)$  è costante nell'intervallo  $t \in [0, \infty)$  se e solo se  $X$  ha distribuzione esponenziale. Questa caratterizzazione viene sfruttata per costruire un test di esponenzialità. Si dimostra la consistenza della statistica test proposta e se ne deriva la distribuzione asintotica. Delle simulazioni indicano che la procedura suggerita ha potenza elevata in una vasta gamma di situazioni.

**Keywords:** Mean residual life, Test of exponentiality, Kolmogorov-Smirnov statistic, Wiener process, Uniform quantile process.

## 1. Introduction

Let  $X$  denote a non negative random variable (r.v.) with continuous distribution function (d.f.)  $F$ . The mean residual life (MRL) at time  $t$  is defined by

$$m(t) = E(X - t|X > t) = \frac{\int_t^\infty \bar{F}(x)dx}{\bar{F}(t)}$$

where  $\bar{F} = 1 - F$  is the survival function. It has been shown by Shanbhag (1970) that  $F$  is exponential with mean  $\theta$  if and only if it holds that the MRL is a constant, i.e.,  $m(t) = \theta, \forall t$ . In this paper we use this characterization in order to provide a test statistic for the hypothesis of exponentiality. Equivalent characterizations have been exploited by Baringhaus and Henze (2000) and Taufer (2000) but our approach in what follows is different.

## 2. The test statistic and its properties

Let  $X_1, \dots, X_{n+1}$  be a random sample from a distribution  $F$  with order statistics  $X_{(1)}, \dots, X_{(n+1)}$  and suppose we wish to test the hypothesis that  $F$  is exponential with unspecified mean  $\theta$  ( $H_0$ ) against a general alternative that it is not exponential. Define the sample MRL after  $X_{(k)}$ , say the  $k^{th}$  failure, as

$$\bar{X}_{>k} = \frac{1}{n - k + 1} \sum_{i=k+1}^{n+1} (X_{(i)} - X_{(k)}) = \frac{1}{n - k + 1} \sum_{i=k+1}^{n+1} (n - i + 2)(X_{(i)} - X_{(i-1)})$$

For convenience denote the normalized spacings

$$Y_i = (n - i + 2)(X_{(i)} - X_{(i-1)}), \quad i = 1 \dots n + 1.$$

Under the null hypothesis of exponentiality, it is known that

$$E(\bar{X}_{>k}) = E(\bar{X}) = \theta, \quad k = 1, 2, \dots, n.$$

The sequence of averages,  $\bar{X}, \bar{X}_{>1}, \dots, \bar{X}_{>n}$  all provide unbiased estimators of the unknown value  $\theta$ . This would suggest the use of a distance measure between these sample means in order to build a test statistic for  $H_0$ . One simple and natural way to do this is to exploit a Kolmogorov-Smirnov type distance, namely, reject  $H_0$  when

$$T_n = \max_{1 \leq k \leq n} \frac{|\bar{X} - \bar{X}_{>k}|}{\bar{X}}$$

is large. Note that the division by the overall sample mean  $\bar{X}$ , makes  $T_n$  scale free. We point out however that  $T_n$ , as it is, does not converge to 0 even under the null hypothesis. This may be immediately seen for example if we note that  $\bar{X}_{>n} = Y_{n+1}$  which, under  $H_0$ , is exponentially distributed with mean  $\theta$  no matter what the sample size is; to see in more detail the behavior of  $T_n$ , let  $S(i) = \sum_{j \leq i} \xi_j$  where  $\xi_j$  are *i.i.d.* exponential r.v.'s with mean 1 and  $i = n - k + 1$  then

$$T_n = \max_{1 \leq i \leq n} \left| 1 - \frac{\bar{X}_{>n-i+1}}{\bar{X}} \right| = \max_{1 \leq i \leq n} \left| 1 - \frac{S(i)}{i} \frac{(n+1)}{S(n+1)} \right| \leq \max_{1 \leq i \leq \infty} \left| 1 - \frac{S(i)}{i} \right| + o_p(1)$$

as  $n \rightarrow \infty$ . These facts may cast some doubt about the efficiency of  $T_n$  in testing for exponentiality. In fact, some power simulations we ran indicated that  $T_n$  has poor power for several alternatives to exponentiality.

Before going further, we may try to connect  $T_n$  with other test statistics proposed in the literature. Note that  $\bar{X}_{>k}$  is the total time on test transform (TTT) after  $X_{(k)}$  divided by  $n - k + 1$ . If we denote the TTT statistics as

$$D_{n+1}(t) = \sum_{i=1}^k Y_i + (n - k + 1)(t - X_{(k)}), \quad t \in [X_{(k)} - X_{(k+1)})$$

then, after some manipulation we can write

$$T_n = \max_{1 \leq k \leq n} \frac{n+1}{n-k+1} \left| \frac{D_{n+1}(X_{(k)})}{(n+1)\bar{X}} - \frac{k}{n+1} \right|$$

One may compare  $T_n$  written in this way with the statistics proposed by Koul (1978) to test against NBUE alternatives and by Baringhaus and Henze (2000) for testing  $H_0$  against omnibus alternatives. The key feature of  $T_n$  is the weight  $(n - k + 1)^{-1}$  which comes out naturally in our approach to the problem where we consider all sample mean residual lives. The question of interest here is if this approach can be more fruitful since as we have seen, the first results on  $T_n$  are not encouraging.

The desire to overcome the problems noted for the test statistics  $T_n$  motivates the construction of a *trimmed* test statistics where some of the last residual means are discarded from  $T_n$ . This has to be done in such a way as to be able to asymptotically estimate  $m(t)$  over the whole real line. We thus define

$$T_n^\alpha = \max_{1 \leq k < n - n^\alpha + 1} \frac{|\bar{X} - \bar{X}_{>k}|}{\bar{X}}, \quad \alpha \in (0, 1).$$

We see that the comparison of the sequence of the residual means goes up to the term with index  $n - [n^\alpha]$  where  $\alpha$  is a parameter which determines the number of 'last' residual means to be discarded and  $[n^\alpha]$  is the greatest integer in  $n^\alpha$ .

To determine the asymptotic properties of  $T_n^\alpha$  note first of all that under  $H_0$ ,  $\bar{X}_{>k} \xrightarrow{a.s.} \theta$ ,  $1 \leq k < n - n^\alpha + 1$  by the strong law of large numbers. From this it follows that, in case of exponentiality  $T_n^\alpha \xrightarrow{a.s.} 0$ . The convergence properties of  $T_n^\alpha$  can be studied under more general conditions, as we do in the following

**Theorem 1.** *Let  $m(t) < \infty$  and  $F$  be a continuous d.f., then*

$$\max_{1 \leq k < n - n^\alpha + 1} |\bar{X} - \bar{X}_{>k}| \xrightarrow{p} \sup_{0 \leq t < \infty} \left| \theta - \frac{\int_t^\infty \bar{F}(x) dx}{\bar{F}(t)} \right|$$

as  $n \rightarrow \infty$ .

The proof is omitted to save space; the techniques used to prove Theorem 1 are similar to those used in Koul (1978). For the asymptotic distribution of  $T_n^\alpha$ , we have the following result

**Theorem 2.** *Let  $\alpha \in (0, 1)$ , then under  $H_0$*

$$n^{\alpha/2} T_n^\alpha \xrightarrow{D} \sup_{0 \leq t \leq 1} |W(t)|$$

where  $W(t)$  is a Wiener process.

Theorem 2 can be proved by using results on the convergence in distribution of supremum functionals of the uniform quantile process in a weighted metric. Such results can be found in Csörgő and Horváth (1993) and details are omitted to save space.

*Remark.* The null distribution of  $T_n^\alpha$  depend on that of an ordered uniform random sample. Hence, as noted by Gupta and Richards (1997) the distribution of  $T_n^\alpha$ , sharing the same invariance property with several other tests for exponentiality, remains the same for all random vectors  $X_1, \dots, X_{n+1}$  having a multivariate Liouville distribution.

### 3. Monte Carlo Power Comparisons

We estimated empirically the power of  $T_n^\alpha$  by generating 10.000 samples of size 10, 30 and 50 for several common alternatives. An excerpt of the results, for tests of size  $\alpha = 0.05$ , is given in Table I. Here we consider the Weibull (W) distribution, linear increasing failure rate distributions (LIFR) with density  $(1 + \theta x) \exp\{-(x + \theta x^2/2)\} \mathbf{1}_{(x \geq 0)}$  and J-shaped (JS) distributions with density  $(1 + \theta x)^{-(\theta+1)/\theta} \mathbf{1}_{(x \geq 0)}$ . We have DFR for Weibull with  $\theta < 1$  and for J shaped distributions; IFR for Weibull with  $\theta > 1$  and for LIFR distributions.

To have an idea of the changing performance of our test statistic by varying the value of the parameter  $\alpha$  we chose the values  $\alpha = 0.4, 0.6, 0.75, 0.9$ , discarding  $[n^\alpha]$  'last' residual means. As a cornerstone for  $T_n^\alpha$ , we reported the power values of the classical Kolmogorov Smirnov statistic (KS) with estimated mean and the statistic  $L_n$  of Baringhaus and Henze (2000). In the last column of the table we report the highest power (HP( $[n^\alpha]$ )) obtained by our test statistic for that distribution and in parenthesis the value of  $[n^\alpha]$  by which this was obtained.

From the results in Table I we note that the choice of  $\alpha$  has a pronounced effect on the power of the test  $T_n^\alpha$ . Note that  $T_n^\alpha$  can have considerably higher power than  $KS$  and  $L_n$

for small samples. The right choice of  $\alpha$  always allow at least to match the power of the other competitors. The same holds slightly to a less extent for moderate or large sample sizes. Also, we note that the choice of  $\alpha$  has generally less effect with increasing sample size obtaining high power for large values of  $\alpha$  in most situations.

**Table I.** Estimated Power for  $T_n^\alpha$ . Tests of size 0.05.

| Alternative | n  | $T_n^{0.4}$ | $T_n^{0.6}$ | $T_n^{0.75}$ | $T_n^{0.9}$ | KS | $L_n$ | HP( $[n^\alpha]$ ) |
|-------------|----|-------------|-------------|--------------|-------------|----|-------|--------------------|
| W(0.6)      | 10 | 44          | 40          | 17           | 2           | 36 | 25    | 44 (4)             |
|             | 30 | 74          | 80          | 83           | 77          | 81 | 78    | 83 (13)            |
|             | 50 | 85          | 92          | 95           | 96          | 96 | 95    | 96 (30)            |
| W(1.6)      | 10 | 10          | 18          | 34           | 37          | 26 | 32    | 37 (7)             |
|             | 30 | 04          | 35          | 66           | 77          | 71 | 78    | 77 (19)            |
|             | 50 | 13          | 48          | 83           | 93          | 92 | 95    | 93 (30)            |
| LIFR(3)     | 10 | 07          | 14          | 22           | 22          | 17 | 22    | 22 (7)             |
|             | 30 | 03          | 25          | 44           | 48          | 43 | 54    | 48 (19)            |
|             | 50 | 10          | 37          | 64           | 70          | 68 | 78    | 70 (30)            |
| JS(1.0)     | 10 | 56          | 49          | 17           | 01          | 49 | 43    | 58 (2)             |
|             | 30 | 92          | 93          | 90           | 85          | 90 | 90    | 93 (8)             |
|             | 50 | 99          | 99          | 99           | 97          | 98 | 98    | 99 (12)            |

On the basis of the results one can try to propose an empirical rule as far as the choice of the number of last residual means to discard is concerned. The clearer situation seems to pertain to the case of IFR (and hence DMRL) distributions. In such a contest a large value of  $\alpha$  gets the highest power values and this happens for sample values of small, moderate and large size. To a certain extent, the contrary happens for DFR distributions, i.e. small values of  $\alpha$  obtain the largest power values however, a wrong choice of  $\alpha$  seems not to have a dramatic effect in moderate or large sample sizes.

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