

Spectral Methods for Fractional ARIMA Processes

Metodi Spettrali per Processi ARIMA Frazionali

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Riassunto: Nel presente lavoro, nella finalità di stimare il parametro di lunga memoria di un processo ARIMA frazionale, ci si avvale di metodologie basate sulla decomposizione ortogonale del processo temporale. In particolare si propone un’integrazione dell’analisi di Fourier con l’espansione di Karhunen-Loève e con la trasformata discreta wavelet al fine di migliorare le proprietà campionarie degli stimatori basati sull’approssimazione della densità spettrale alle basse frequenze.

Keywords: Long Memory Temporal Processes, Karhunen Loève Expansion, Fourier Analysis, Discrete Wavelet Transform.

1. Fractional ARIMA processes

A temporal process $\{X(t), t \in \mathbb{N}\}$ is said fractional if the autocorrelation coefficients are not absolutely summable. This condition (Beran 1994) is equivalent to the spectral density function $f(w)$ having a pole at the origin of the form

$$f(w) \sim a|w|^{1-2H} \quad w \rightarrow 0 \quad (1)$$

where a is a positive constant and H is the Hurst exponent. For a stationary and invertible long-range dependency process the Hurst parameter is included in the range $(0,1)$. In particular if $0 < H < 0.5$ the process is said to be antipersistent; if $H = 0.5$ the process is either a short memory process or a white noise and, finally, if $0.5 < H < 1$ the process exhibits long memory. Among the stationary, invertible and long memory processes the most known ones are the ARFIMA $_{(p,d,q)}$ (Beran 1994)

$$f(B)(1-B)^d X(t) = q(B)e(t) \quad (2)$$

where B is the backward operator, $f(B)$ is the characteristic autoregressive polynomial, $q(B)$ is the moving average characteristic polynomial, $e(t)$ is a Gaussian white noise and $d = H - 0.5$ is the fractional integration parameter varying in the interval $(-0.5, 0.5)$.

In this paper the aim is to exploit the spectral features of long memory processes to estimate the Hurst parameter given the observed data. The spectral decomposition of the process is considered both in the frequency domain and in the temporal one. The spectral methods of Karhunen Loève and Wavelet Transform are used to improve the performances of the estimators based on equation (1) when the process contains both long memory and short memory components.

2. Spectral decomposition

If $\{X(t), t \in T\}$ is a finite energy process, it is possible to expand it in a set of deterministic functions $f_n(\cdot)$, $n \in N$, which form a complete orthogonal system in the space of square integrable functions $L_2(T)$

$$X(t) = \sum_{n \in N} a_n f_n(t) \quad (3)$$

where the stochastic coefficients a_n , $n \in N$ are given by the following inner product

$$a_n = \int_T X(t) f_n(t) dt \quad n \in N \quad (4)$$

Fourier decomposition, in the frequency domain, *Karhunen Loève expansion (KLE)*, in the time domain, and *discrete orthogonal wavelet*, in the frequency-time domain, are special cases of equation (3). In the Fourier analysis (Priestly 1981) the process is expressed in the system formed by the frequency depending exponential functions e^{itw} , $w \in [-p, p]$, and the Fourier coefficients (4) give information on the spectral contents of the process. In the time domain the KLE allows to represent a zero mean random process $X(t)$ as a series of orthogonal functions with uncorrelated, zero mean coefficients given by the *Karhunen Loève transform (KLT)* (4) (Basilevsky 1994). The basis functions $f_n(\cdot)$, $n \in N$, are the eigenvectors of the covariance matrix. The variance of the coefficients is equal to the corresponding eigenvalues. Therefore the eigenvalues, known as generalized spectrum, give information on the amount of process variance explained by each eigenvector. The series expansion (3) of the process in an orthogonal basis can be realized also in the wavelet domain (Mallat 1998). We focus our attention only on the *Discrete Wavelet Transform (DWT)*. In this case the construction of the basis of the Hilbert space $L_2(T)$ is obtained by translating the father wavelet $j(\cdot)$ (scaling function) and by translating and dilating the mother wavelet $y(\cdot)$ (analysing function). Consequently the orthonormal basis $\{j_{j_0,k}(\cdot), y_{j,k}(\cdot)\}_{j \geq j_0, k \in Z}$, for some fixed $j_0 \in Z$, is given by

$$j_{j_0,k}(t) = 2^{j_0/2} j(2^{j_0} t - k) \quad y_{j,k}(t) = 2^{j/2} y(2^j t - k) \quad (6)$$

In the wavelet basis, equation (3) can be expressed as

$$X(t) = \sum_{k \in Z} c_{j_0,k} j_{j_0,k}(t) + \sum_{j \geq j_0} \sum_{k \in Z} d_{j,k} y_{j,k}(t) \quad (7)$$

where the scaling, $c_{j_0,k}$, and the wavelet, $d_{j,k}$, coefficients are given by

$$c_{j_0,k} = \int_T X(t) j_{j_0,k}(t) dt \quad d_{j,k} = \int_T X(t) y_{j,k}(t) dt \quad (8)$$

3. Long memory parameter estimation in the spectral domain

The estimation of the long memory parameter H in the spectral domain exploits the features of the spectral density function near the origin. Given equation (1) and using the periodogram as natural estimator of the spectral density, the linear regression of the log periodogram on the log frequencies yields to a heuristic estimator of the Hurst parameter (Beran 1994). A semi parametric estimator based on (1) and proposed by Geweke and Porter-Hudack (GPH) (1983), considers the following regression for $w \in \mathbb{R}$

$$\text{Log}[f(w)] = \text{Log}[s^2 f_A(0)/2p] - (H - 0.5) \text{Log}[4\sin^2(w/2)] + \text{Log}[f_A(w)/f_A(0)] \quad (9)$$

where s^2 is the process variance and $f_A(w)$ is the spectral density function of the stationary, invertible and short memory process $A(t) = [\mathbf{q}(B)\mathbf{f}^{-1}(B)]\mathbf{e}(t) = (1-B)^d X(t)$. Equation (9) holds under the hypothesis that $f_A(w)$ is finite for $w=0$. Since the simultaneous presence of long and short memory components might yield to biased estimates of the Hurst parameter, it could be convenient to detect the components according to their temporal scale of variability. Ordering the eigenvectors according to non increasing magnitude of the corresponding eigenvalues, it is well known (Basilevsky 1994) that the first eigenvectors are associated with the large scale variability, the following are associated with variability of decreasing scale and, finally, the last ones are related to the noise component. The explorative analysis of the eigenvectors of the covariance matrices of simulated time series from $\text{ARFIMA}_{(0,d,0)}$ processes highlights that the eigenvectors associated to the largest eigenvalues show the spectral features of long memory processes while the remaining are related to the noise. Unfortunately the eigenvectors obtained from realizations of $\text{ARFIMA}_{(p,d,q)}$ processes do not show a so clear pattern since the short-range components contaminate their behaviour. Two different techniques based on the KLE are proposed in order to estimate the fractional parameter. In the first approach, the autoregressive and moving average parameters are estimated from the time series reconstructed applying the KLE just to the eigenvectors which do not exhibit “long memory” features. In order to select the eigenvectors related to the large scale variability, the Lobato Robinson (1998) test is applied to each eigenvector. Once estimated the short memory parameters, the Hurst exponent estimate might obtained by means of both the regression on the log-log periodogram and the GPH estimator using the residual series given by

$$F(t) = [1 - \hat{f}(B)]X(t) + \hat{q}(B)\mathbf{e}(t) \quad (10)$$

The proposed methodology has been applied to time series simulated from the $\text{ARFIMA}_{(p,d,q)}$ processes for different combination of the long memory and short memory parameters. In general with respect to the use of the two estimators directly on the simulated time series, this procedure leads to a reduced mean square error. In particular this improvement is due principally to the reduction of the bias. Furthermore the sample distribution of both estimators is less skewed and more similar to the Gaussian distribution.

In the second approach the parameter H is directly estimated from the reconstructed long-range dependency component. As noted, for $\text{ARFIMA}_{(p,d,q)}$ processes the eigenvectors do not behave as those of simple $\text{ARFIMA}_{(0,d,0)}$, but are affected by short

memory contamination. To overcome this limitation, we propose to filter the observed data by applying a linear approximation in the wavelet domain. Hence, chosen a suitable level of approximation j_0 , the filtered signal is given by the first summation of equation (7). The analysis of the eigenvectors of the signal covariance matrix stresses the opportunity of a dimensionality reduction. In fact just the first ordered eigenvectors give a significant contribution to the process variability explanation. This conclusion is confirmed by the convergence of both the log-log periodogram and the GPH estimates of H as the number of eigenvectors used in reconstruction (3) increases. When convergence is achieved the definitive estimates are obtained. As in the previous approach, this methodology has been tested on the same simulated time series. As before an improvement in terms of bias has been noted, even if the increasing variability of the estimates yields to some increments in the mean square error. In any case, an important result to stress for both procedures is a reduction of the risk to get Hurst exponent estimates outside the range (0.5, 1). The proposed methodologies have been applied to the well-known Nile River Minima yearly time series for the years 622-1281 (Beran 1994).

Table 1: Estimates of the Hurst exponent H for the Nile River Minima time series

	Series	Log-log periodogram			GPH		
		Raw time series	KLE	KLE & DWT	Raw time series	KLE	KLE & DWT
H estimates		0.9353	0.9093	0.9192	0.9932	0.9772	0.9838
Mean	1147.8	1147.2	1147.4	1147.2	1147.3	1147.5	1147.3
Standard deviation	91.08	62.23	78.53	61.02	66.19	82.61	65.68
Kurtosis	0.0820	-0.2927	-0.0283	-0.2926	-0.2886	0.0324	-0.2896
Skewness	0.2741	0.0874	0.2415	0.0844	0.0999	0.2583	0.0978
Range	531	316.8	439.5	310.2	338	475	334.8
R-square		0.4524	0.4210	0.4627	0.4528	0.4132	0.4631

The comparison between the observed time series and the fitted ones for the different estimates highlights an improvement of the R-square for of the approach based on the integration of the KLE and the wavelet procedure. In contrast the first approach, based just on the KLE, yields to better global statistics. In particular the fitted series, obtained by means of this approach, are more similar to the original time series in terms of kurtosis and skewness.

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