

# Bootstrapping and Bartlett corrections in cointegrated VARs <sup>1</sup>

## *Bootstrap e correzioni di Bartlett nei modelli VAR cointegrati*

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**Riassunto:** I test sui coefficienti di cointegrazione costituiscono ancora un serio problema nelle applicazioni empiriche. Per piccoli campioni la procedura asintotica di Johansen (1991) è seriamente distorta; le possibili alternative sono: (a) correzioni di Bartlett; (b) test bootstrap. In questo lavoro riesaminiamo le modalità sia del calcolo del fattore di Bartlett che di costruzione di test bootstrap. Simulazioni Monte Carlo portano alle seguenti conclusioni: (1) i test asintotici sono talmente distorti che non dovrebbero mai essere usati; (2) i test corretti mediante il fattore di Bartlett, pur avendo errori di prima specie molto maggiori della significatività nominale, sono comunque molto meno distorti delle procedure asintotiche; (3) il test bootstrap proposto ha la minore distorsione con potenza solo leggermente minore dei test con correzione di Bartlett.

**Keywords:** Cointegration, Bootstrap, Bartlett correction.

## 1. Introduction

The small sample properties of tests on long-run coefficients in cointegrated systems are still a matter of concern to applied econometricians. Since the asymptotic procedures proposed by Johansen (1991) have been shown to suffer from severe size distortion, two natural and complementary solutions have been proposed: (i) applying Bartlett corrections to the test statistics, in the hope that the corrected statistic will follow a small sample distribution closer to the asymptotic one, and thus bring actual sizes closer to the nominal sizes (Johansen, 2000); (ii), trying to estimate the actual small sample distribution by the bootstrap, a computer-intensive technique strictly linked with the Edgeworth expansion (Fachin, 2000, Gredenhoff and Jacobson, 2001, and references therein). For the time being, no definite solution has however appeared. The primary aim of this paper is precisely that of comparing size and power properties of the bootstrapped tests on the coefficients of the cointegration relations.

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## 2. Bartlett-corrected and Bootstrap Tests on Cointegrating Coefficients

The idea behind the Bartlett correction is both simple and appealing. Suppose the aim is testing an hypothesis on a subset  $\theta$  of the parameters  $\Theta$ ,  $H_0 : \theta = \theta^0$ . In regular cases, the LR test statistic  $S$  has an expected value of  $E[-2 \ln(LR)] = h \left(1 + \frac{1}{T}g(\theta)\right) + O\left(\frac{1}{T^2}\right)$ , where  $h$  denotes the number of restrictions tested. Then dividing the test statistic  $S$  by  $\left(1 + \frac{1}{T}g(\theta)\right)$  we may obtain the modified test statistic  $S_B$  and expect the resulting distribution to be closer to a  $\chi^2$  distribution. This division is called a Bartlett correction and  $\frac{1}{T}g(\theta)$  will be referred to as the Bartlett factor. Johansen (2000) derived the Bartlett correction for three different kind of hypotheses on  $\beta$ : (1)  $\beta = \beta^0$ , a simple hypothesis on all the cointegration vectors; (2)  $\beta_1 = \beta_1^0$  where  $\beta_1^0$  are the first  $r_1$  relations ( $1 \leq r_1 < r$ ) and the other cointegration relations are unrestricted; (3)  $\gamma = H\varphi$  where  $H$  is a  $(r \times s)$  matrix of full rank and  $s < r$ . This hypothesis implies the same restriction on all relations in  $\gamma$ . The problem with the Bartlett correction is that it is extremely difficult to derive. At the opposite, the great advantage of the bootstrap is that in principle it can offer immediate solutions to new problems. However, in practice its ability to deliver good alternatives when reliable small sample parametric procedures are lacking must be accurately tested before its use may be recommended. The general idea underlying bootstrap tests is to assess the value of the test statistic  $s$  obtained from the empirical analysis on the basis of the distribution of a large number of statistics  $s^*$  computed from suitably constructed pseudodata, with the null hypothesis of the former consistent with the data generating process (DGP) of the latter. To this end,  $H_0$  may be imposed when generating the pseudodata, or, vice versa, the chosen DGP taken as the null hypothesis (as recommended by Hall, 1992). In both cases,  $H_0$  is true for the pseudodata, and thus, assuming for simplicity a one-sided test, the proportion of  $s^*$  more extreme than  $s$  in the relevant direction is a natural estimate of the  $p$ -value of the test. With cointegrated VARs and some hypothesis on the long-run coefficients  $H_0 : \beta = \beta^0$ , the two approaches entail respectively: (a) estimating a VAR constrained under  $H_0 : \beta = \beta^0$ , generating the pseudodata on the basis of the estimated *constrained* coefficients and a set of random noises (we will discuss the choice of these below), and testing  $H_0 : \beta = \beta^0$  both on the original data and on the pseudodata; (b) estimating an unconstrained VAR, generating the pseudodata on the basis of the estimated *unconstrained* coefficients and a set of random noises, testing  $H_0 : \beta = \beta^0$  on the original data and  $H_0^* : \beta = \hat{\beta}$  (where  $\hat{\beta}$  are the unconstrained estimates of  $\beta$ ) on the pseudodata. So far, approach (a) has been favoured with no exception in the applications of interest here. However, a point of crucial importance for testing in the maximum likelihood estimation of cointegrated VARs seems to have gone unnoticed: although both approaches are valid and asymptotically equivalent under  $H_0$ , this is not true any more when it is false<sup>2</sup>. Thus, we will consider bootstrap tests of type (b). Defining  $\Theta$  the entire parameter set of the VAR and assuming we are interested in running a one-sided test on a subset  $\theta$ , with  $H_0 : \theta = \theta^0$ , the general structure of the bootstrap test we shall implement

<sup>2</sup>To see this, consider the case of a test  $H_0 : \beta = \beta^0$  in a model without lags and just one cointegration vector. If this vector is misspecified, then  $\beta^{0'} X_{t-1}$  is clearly an  $I(1)$  process, whereas  $\Delta X_t$  is  $I(0)$ . The only congruent values for the loading factors  $\alpha$  are therefore zero. Hence all the element of the matrix  $\hat{\Pi} = \hat{\alpha}\beta^{0'}$  equal zero (asymptotically) and the rank of such a matrix is 0 not 1. If one were to use this matrix for the Bootstrap DGP, one would generate just random walks without any cointegration.

is thus the following:

(1) Estimate VAR on data  $\mathbf{X}$ ; for given cointegrating rank obtain estimates  $\widehat{\Theta}$ , residuals  $\widehat{\varepsilon}$ , and test statistic  $s$  for the hypothesis  $H_0 : \theta = \theta^0$ ; (2) Construct pseudodata:  $X^* = \phi(\widehat{\Theta}, \varepsilon^*)$ ,  $\varepsilon^*$  drawn at random with replacement from  $\widehat{\varepsilon}$  or  $NID$ . (3) Estimate VAR on pseudodata  $X^*$ ; obtain coefficients  $\widehat{\Theta}^*$  and test statistic  $s^*$  for the hypothesis  $H_0^* : \theta = \widehat{\theta}$ ; (4) Repeat 2-3 a large number of times; (5) Compute bootstrap  $p$ -value:  $p^* = \text{prop}(s^* > s)$ .

As mentioned in the introduction, Davidson and MacKinnon (2000) recently put forth a computationally cheap double bootstrap procedure which may deliver results superior to the standard bootstrap just outlined, and which we shall thus examine.

### 3. Design of the Monte Carlo Experiment

The key characteristics of the DGP to be controlled in the experiments are the dimension of the system, i.e. number of variables and lags, and its long-run structure, i.e. number of the cointegrating relationships and the speed at which the system adjusts to them. The experimental design adopted here will thus generalize to a multivariate system the classical DGP used by a number of studies, which allows an easy control of the speed of adjustment. We shall consider systems including  $p = 5$  random variables and with  $r = 1$  or 2 cointegrating relationships. Let  $\mathbf{x}_t = [x_{1t} \dots x_{pt}]'$  be the column vector of the realizations of the random variables of interest at time  $t = 1, \dots, T$ ,  $\mathbf{u}_t = [u_{1t} \dots u_{pt}]'$  the errors,  $\epsilon_t = [\varepsilon_{1t} \dots \varepsilon_{pt}]'$  the noise, whose stochastic structure will be discussed in detail below, and  $\tau$  a time trend. Our DGP is then given by  $\mathbf{G}\mathbf{x}_t + \rho\tau = \mathbf{u}_t$ , with  $\Phi\mathbf{u}_t' = \epsilon_t$ ,  $\mathbf{G} = [\gamma_1 \dots \gamma_r]'$ ,  $\gamma_j = [\gamma_{j1} \dots \gamma_{jk}]$ ,  $\Phi = \text{diag}(\phi)$ ,  $\phi = [\phi_1(L) \quad \phi_2(L) \quad \phi_3(L) \quad \phi_4(L) \quad \phi_5(L)]$ ,  $\sum \phi_i = 0.7$ . Given that we are using a full-information method we do not need to worry about endogeneity; we shall thus consider a very simple structure, with one stochastic trend ( $X_p$ ) transmitted to the first  $r$  variables of the system, while the remaining  $p - r - 1$  follow independent random walks. The number of both Monte Carlo replications and bootstrap redrawings has been fixed to 500: on the basis of previous work and some pilot experiments we concluded that the gain in precision deliver by higher numbers of either was not worth the higher computing costs and longer calendar time required. At 0.05 the Monte Carlo standard error will thus be about 0.010. Finally, for space constraints we will report only a summary table. A full set of results is available on request.

### 4. Conclusions

We have compared different variants of bootstrap and Bartlett-corrected tests in a DGP which is relatively unfavourable, but reproduces some features of real life empirical applications: a relatively large system (5 variables and 2 or 4 lags), and rather slow adjustment to long-run equilibrium. With such a complex DGP the caveats common to all simulation studies are even more important than usual. Further, the type of tests examined assumes full knowledge of the tested cointegrating vectors, a rare event in practice: however, they are the only tests for which the Bartlett correction is available. With all these

caveats, our recommendations are the following: (i) Asymptotic tests should be used in no circumstance; (ii) Bartlett-corrected tests may be used provided considerable caution is exercised, as their Type I error is often much larger than the nominal size; (iii) Bootstrap tests, with a somehow lower size distortion than the Bartlett corrected tests accompanied by limited power losses, may also be used; the fast double bootstrap delivers the best performance, and thus it appears to be a powerful tool for applied work, especially in the many cases when the Bartlett correction is not available.

**Size and Power: summary results**  
*1 to 2 cointegration vectors, test on 1 to 2 vectors*

| <i>rank, tested vectors</i> | 1,1         |              | 2,1         |              | 2,2         |              |
|-----------------------------|-------------|--------------|-------------|--------------|-------------|--------------|
|                             | <i>size</i> | <i>power</i> | <i>size</i> | <i>power</i> | <i>size</i> | <i>power</i> |
| Asymptotic                  | 66.0        | 99.0         | 39.2        | 97.6         | 68.6        | 98.2         |
| Bartlett                    | 35.8        | 92.2         | 15.8        | 79.2         | 33.2        | 82.2         |
| Bootstrap                   | 32.0        | 86.0         | 15.2        | 77.2         | 28.2        | 74.6         |
| FDBS                        | 26.2        | 76.0         | 13.4        | 68.0         | 20.0        | 62.2         |

nominal significance level: 5%; FDBS: Fast Double Bootstrap. Power simulations:

case (1,1):  $H_0 : \beta_1^0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \end{bmatrix}$ , DGP:  $\beta_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0.5 \end{bmatrix}$

case (2,1): as case (1,1) with DGP:  $\beta_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \end{bmatrix}$

case (2,2): as case (2,1) with  $H_0 : \beta_2^0 = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \end{bmatrix}$

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