

Estimating unobservable signal by Markovian noise induction

*Stima di un segnale non osservabile
attraverso una perturbazione markoviana*

Stefano M. Iacus

Dipartimento di Economia
Università di Milano
stefano.iacus@unimi.it

Ilia Negri

Dipartimento di Ingegneria
Università di Bergamo
negri@unibg.it

Riassunto: Un segnale non osservabile viene trasmesso attraverso un canale ma può essere rilevato solo quando viene aggiunto ai dati un rumore con struttura nota e proporzionale ad un prestabilito livello. Il livello ottimale di rumore, cui corrisponde la massima informazione di Fisher del modello, viene detto *di risonanza stocastica*. L'effetto di risonanza stocastica è stato osservato in presenza di differenti strutture del rumore. Qui viene studiato il caso in cui il rumore è rappresentato da un processo markoviano. Vengono proposti stimatori consistenti del segnale non osservabile e viene analizzato un problema di verifica di ipotesi. L'effetto della risonanza stocastica è evidenziato, tramite esempi, sia nel problema di stima che in quello della verifica di ipotesi.

Keywords: stochastic resonance, Markov processes, unobservable signal detection, maximum a posterior probability.

1. Introduction

The term “stochastic resonance” was introduced in the early '80s (see Benzi *et al.* (1981) and Nicolis (1982)) in the study of periodic advance of glaciers on Earth. The stochastic resonance is the effect of non-monotone dependence of the response of a system on the noise when this noise (for instance the temperature) is added to a periodic input signal (see e.g. Klimontovich (1999), in which the author explains also differences and similarities with the notion of stochastic filtering). An extensive review on stochastic resonance and its presence in different fields of applications can be found in Gammaitoni *et al.* (1998). Following Chapeau-Blondeau and Rojas-Varela (2000), as stochastic resonance we intend the phenomenon in which the transmission of a signal can be improved (in terms of statistical quantities) by the addition of noise. From the statistical point of view the problem is to estimate a signal $\{f(t) : t < T\}$ transmitted through a channel. This signal has to be detected by a receiver that can reveal signals louder than a threshold τ . If $f(\cdot)$ is bounded from above by τ , the signal is not observable and the problem has not a solution. But, if some noise $\{\varepsilon(t); t < T\}$ is added to the signal, the perturbed signal $y(t) = f(t) + \varepsilon(t)$ may be observable and inference can be done on $f(\cdot)$. Too few noise is not sufficient to give good estimates and too much noise deteriorates excessively the signal. The optimal -in some sense- level of the noise will be called stochastic resonance in this framework. Usually (see Gammaitoni *et al.* (1998)) the criterion applied to measure optimality of estimators are the Shannon mutual information or the Kullback divergence. More recently the Fisher information quantity have been also proposed (see Chapeau-Blondeau and Rojas-Varela (2000), Greenwood *et al.* (2000), Iacus (2001)). He-

re we are concerned with the Fisher information quantity. It happens that this quantity, as a function of the noise, can be maximized for certain noise structures. If there is only one global maximum, the corresponding noise level is the value for which we have stochastic resonance, if several local maxima are present, the phenomenon is called stochastic multi-resonance.

2. Model specification and main results

We study the problem of estimation and hypotheses testing for the following model: we suppose to have a threshold $\tau > 0$ and a subthreshold constant and non negative signal $f(t) = \theta, \theta < \tau$. We add, in discrete time, a noise that is a trajectory of an ergodic Markov process $\{X_{t_n} : n \geq 1\}$ and we observe the perturbed signal

$$Y_{t_i} = \theta + X_{t_i}, \quad i = 1, \dots, n$$

only when it exceeds the threshold τ , where $\varepsilon > 0$ is the level of the noise. We are interested in studying estimation problems as n goes to infinity. Thus we introduce consistent and asymptotically efficient estimators that are linked with the ergodic distribution of the Markov process.

In order to derive estimators easy to calculate, we will consider several schemes of observations, i.e. we record the time that the noisy signal exceeds the threshold or we measure the energy of the partially observable signal. The first scheme of observation, known as time-integrating detector has been fruitfully used in the case of independent and identically distributed and time-equally spaced shot noises by Müller and Ward (2000) and in the case of continuous-time ergodic diffusion noise induction by Iacus (2001).

The observed process can be written as follows

$$Z_{t_i} = Y_{t_i} \cdot \mathbf{1}_{\{Y_{t_i} > \tau\}}, \quad i = 1, \dots, n,$$

where $\mathbf{1}_A$ is the indicator function of the set A . Let

$$p_i = P(Y_{t_i} > \tau)$$

be the probability of observing the noisy signal at time t_i , then it can be rewritten in terms of the law of the process $\{X_{t_i} : i = 1, \dots, n\}$ as follows

$$p_i = 1 - P\left(X_{t_i} < \frac{\tau - \theta}{\varepsilon}\right) = 1 - F_i\left(\frac{\tau - \theta}{\varepsilon}\right)$$

where $F_i(\cdot)$ is the distribution function of X_{t_i} , $i = 1, \dots, n$. As n goes to infinity, F_i converges to the ergodic distribution function F of the Markov process. So, the parameter admits the following representation

$$\theta = \tau - \varepsilon \cdot F^{-1}(1 - p) + o_n(1),$$

where $p = 1 - F\left(\frac{\tau - \theta}{\varepsilon}\right)$. Thus, a natural estimator for θ is

$$\hat{\theta}_n = \tau - \varepsilon \cdot F^{-1}(1 - \hat{p}_n)$$

where $\hat{p}_n$ is the proportion of time spent by the process $\{Y_y\}$ over the threshold with respect to the whole time interval $[t_0, t_n]$. From the above, it is clear that there is an effect of the level of the noise ε on the asymptotic variance of the estimator $\hat{\theta}_n$. We show that this is a non-linear dependency, as to say, the model exhibits stochastic resonance.

The estimator $\hat{p}_n$ is called the time-integrator in the signal processing literature and it is just a transformation of the empirical distribution function estimator of F evaluated at $(\tau - \theta)/\varepsilon$, i.e.

$$\hat{p}_n = 1 - \hat{F}_n\left(\frac{\tau - \theta}{\varepsilon}\right)$$

It is clear that $\hat{p}_n$ is a consistent and asymptotically normal estimator of p by the central limit theorem for the empirical distribution function for Markov processes as proved by Greenwood and Wefelmeyer (1995) for Markov chains and by Negri (1998) for diffusion processes.

Similar results hold when we measure the energy of the partially observed signal i.e.

$$\nu_n = \frac{1}{n} \sum_{i=1}^n Z_{t_i}^2.$$

It is also possible to show that the derived estimators are linked with the maximum likelihood estimators for each scheme of observation, and thus optimality properties can also be inferred from this fact. For the seek of completeness we will derive the asymptotic variance of the estimators by using the so called δ -method as detailed in Barndorff-Nielsen and Cox (1989).

For the same model we also solve the problem of testing the simple hypothesis $\theta = \theta_0$ against the simple alternative $\theta = \theta_1$ by applying the bayesian maximum a posterior probability criterion. It emerges that the overall probability of error is non-monotonically dependent on ε . We show again that there exists a non trivial local minimum of this probability that is again the effect of stochastic resonance.

3. Examples and further extensions of the model

We show several examples where the optimal level of the noise, the stochastic resonance level, is calculated explicitly.

Moreover, it is not assumed in our model that the instant of the noisy shocks $\{t_n, n \geq 0\}$ should be equally spaced as is usually done in the literature (see e.g. Müller and Ward (2000) and Greenwood *et al.* (2000)).

The problem of a parametric non constant signal can also be treated while the full nonparametric non constant signal requires more attention and will be object for further investigations. For i.i.d. observations, Müller (2000) and Müller and Ward (2000) considered the problem of non parametric estimation for regression models of the form $Y(t_i) = s(t_i) + \sigma(t_i)$, $i = 1, \dots, n$. Their approach can be applied in this context.

Other criterion of optimality than the Fisher information quantity can be used as it is usually done in information theory (e.g. Shannon mutual information or Kullback divergence).

The analysis of the overall probability of error seems to put in evidence something new with respect to the current literature (see e.g. Chapeau-Blondeau (2000)).

Riferimenti bibliografici

- Barndorff-Nielsen O. and Cox D.R. (1989) *Asymptotic techniques for use in statistics*, Chapman & Hall, Bristol.
- Benzi R., Sutera A. and Vulpiani A. (1981) The mechanism of stochastic resonance, *J. Phys. A*, 14, 453–458.
- Chapeau-Blondeau F. (2000) Nonlinear test statistic to improve signal detection in non-gaussian noise, *IEEE Sig. Proc. Lett.*, 7, 7, 205–207.
- Chapeau-Blondeau F. and Rojas-Varela J. (2000) Information-theoretic measures improved by noise in nonlinear systems, *Proc. 14th Int. Conf. on Math. Theory of Networks and Systems, Perpignan, France*, 79–82.
- Gammaitoni L., Hänggi P., Jung P. and Marchesoni F. (1998) Stochastic resonance, *Rev. of Modern Phys.*, 70, 223–288.
- Greenwood P.E., Ward L.M. and Wefelmeyer W. (2000) Statistical analysis of stochastic resonance in a simple setting, *Phys. Rev. E*, 60, 4, 4687–4695.
- Greenwood P.E. and Wefelmeyer W. (1995) Efficiency of empirical estimators for markov chains, *Ann. Statist.*, 23, 132–143.
- Iacus S.M. (2001) Statistical analysis of stochastic resonance with ergodic diffusion noise, *Stochastics and Stochastics Reports*, to appear.
- Klimontovich Y.L. (1999) What are stochastic filtering and stochastic resonance?, *Physics-Uspeky*, 42, 1, 37–44.
- Müller U.U. (2000) Nonparametric regression for threshold data, *Canadian J. Statist*, 28, to appear.
- Müller U.U. and Ward L.M. (2000) Stochastic resonance in a statistical model of a time-integrating detector, *Phys. Rev. E*, 61, 3, to appear.
- Negri I. (1998) Stationary distribution function estimation for ergodic diffusion process, *Stat. Inf. for Stoch. Processes*, 1, 61–84.
- Nicolis C. (1982) Stochastic aspects of climatic transitions—response to a periodic forcing, *Tellus*, 34, 1, 1–9.