

A Noise Estimation Method for Corrupted Correlated Data

Un Metodo di Stima dell'Errore di Misura per Dati Correlati

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Riassunto: La rilevazione dei dati, sia nel dominio del tempo che in quello dello spazio, dà luogo in molti casi ad osservazioni caratterizzate dalla presenza di un errore di misura. Una stima di tale componente, che impedisce l'osservazione del fenomeno oggetto di studio (segnale), deve necessariamente precedere l'identificazione del modello di interesse. Obiettivo del presente lavoro è quello di proporre un metodo di stima dei momenti della componente erratica quando si disponga di due rilevazioni dello stesso fenomeno.

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1. Introduction

In almost every fields of science the data acquisition is subject to imperfections due to several possible external sources of error. In statistics this undesired component of the observations is known as *measurement (observation) noise*, and is usually modelled as an additive zero mean process uncorrelated with respect to the true signal. Let $X(s)$ be the signal of interest and $\varepsilon(s)$ the measurement noise, then the general model for the observed variable $Y(s)$ is

$$Y(s)=X(s)+ \varepsilon(s) \tag{1}$$

Note that in this general formulation s is a k -dimensional index vector (e.g. $k=1$ in time series analysis and $k=2$ or 3 in spatial analysis).

Since the presence of $\varepsilon(s)$ strongly affects the parametric estimation of the model chosen for $X(s)$, the goal of this paper is to propose an estimation method of the noise covariance matrix.

2. Noise Covariance Matrix Estimation

In this section the correlation theory is used to estimate the noise autocovariance matrix from two observations of the same phenomenon. This information is necessary for the process of filtering which is strongly affected by the presence of noise. In the statistical literature several methods have been proposed for noise estimation and most of them assume Gauss-distributed noise (Olsen, 1993). As in our case, other methods, above all in an image analysis context, determine the noise statistics from two realizations of the same image (Sijbers *et al.*, 1998). In this article we propose a general “distribution-free” method which is suitable for spatial and temporal series. In what follows we present the method for the one dimension case but the extension to higher dimensions cases is straightforward. Let us assume that two zero mean temporal processes $\{Y(t) : t=1,2,\dots,T\}$ and $\{Z(t) : t=1,2,\dots,T\}$ are characterised by the same common signal $\{X(t) : t=1,2,\dots,T\}$ and different noise components $\{\xi(t) : t=1,2,\dots,T\}$ and $\{\eta(t) : t=1,2,\dots,T\}$. Given the two processes, we rewrite model (1) as

$$Y(t)=X(t)+ \zeta(t) \quad (2)$$

$$Z(t)=X(t)+ \eta(t) \quad (3)$$

Under weakly stationary conditions and provided that the signal and noise are uncorrelated, the autocovariance functions $\gamma_{yy}(h)$ and $\gamma_{zz}(h)$ of the two processes are

$$\begin{aligned} \gamma_{yy}(h) &= E[Y(t+h) Y(t)] \\ &= E[(X(t+h)+ \zeta(t+h)) (X(t)+ \zeta(t))] \\ &= \gamma_{xx}(h) + \gamma_{x\zeta}(h) + \gamma_{\zeta x}(h) + \gamma_{\zeta\zeta}(h) \\ &= \gamma_{xx}(h) + \gamma_{\zeta\zeta}(h) \end{aligned} \quad (4)$$

$$\begin{aligned} \gamma_{zz}(h) &= E[Z(t+h) Z(t)] \\ &= E[(X(t+h)+ \eta(t+h)) (X(t)+ \eta(t))] \\ &= \gamma_{xx}(h) + \gamma_{x\eta}(h) + \gamma_{\eta x}(h) + \gamma_{\eta\eta}(h) \\ &= \gamma_{xx}(h) + \gamma_{\eta\eta}(h) \end{aligned} \quad (5)$$

where $\gamma_{xx}(h)$, $\gamma_{\eta\eta}(h)$ and $\gamma_{\zeta\zeta}(h)$ are the autocovariance functions of the signal and of the two measurement noise processes. More over, taking the cross-covariance function of the processes $Y(t)$ and $Z(t)$, we have

$$\begin{aligned} \gamma_{yz}(h) &= E[Y(t+h) Z(t)] \\ &= E[(X(t+h)+ \zeta(t+h)) (X(t)+ \eta(t))] \\ &= \gamma_{xx}(h) + \gamma_{\zeta x}(h) + \gamma_{x\eta}(h) + \gamma_{\zeta\eta}(h) \end{aligned} \quad (6)$$

$$\begin{aligned}
\gamma_{zy}(h) &= E[Z(t+h) Y(t)] \\
&= E[(X(t+h) + \eta(t+h)) (X(t) + \zeta(t))] \\
&= \gamma_{xx}(h) + \gamma_{x\zeta}(h) + \gamma_{\eta x}(h) + \gamma_{\eta\zeta}(h)
\end{aligned} \tag{7}$$

Assuming that the observation noise $\zeta(t)$ and $\eta(t)$ are mutually uncorrelated, it follows that $\gamma_{yz}(h) = \gamma_{xx}(h)$ and $\gamma_{zy}(h) = \gamma_{xx}(h)$. As it can be noted, equation (6) and (7) give two estimators of the same parameter function $\gamma_{xx}(h)$. Of course this does not constitute a matter when the aim is to estimate the noise variance; in fact, for $h=0$ effectively $\gamma_{zy}(0) = \gamma_{yz}(0)$. On the other hand, when the estimate of the noise covariance matrix is required, a practical solution could be that of taking $\gamma_{xx}(h)$ as

$$\gamma_{xx}(h) = \frac{1}{2} [\gamma_{yz}(h) + \gamma_{zy}(h)] \tag{8}$$

Finally, by substituting (8) in equations (4) and (5) we may estimate the noise autocovariance functions as

$$\gamma_{\zeta\zeta}(h) = \gamma_{yy}(h) - \frac{1}{2} [\gamma_{yz}(h) + \gamma_{zy}(h)] \tag{9}$$

$$\gamma_{\eta\eta}(h) = \gamma_{zz}(h) - \frac{1}{2} [\gamma_{yz}(h) + \gamma_{zy}(h)] \tag{10}$$

3. Simulation Studies

The theoretical results obtained in the previous section are now applied to demonstrate, by experimental simulations in one and two dimensions, how the noise variances can be estimated (setting $h=0$ in (9) and (10)) from two realisations of the same phenomenon. Note that although uncorrelated noises are considered in the experiments, the proposed methodology also allows to deal with correlated noise by estimating the entire autocovariance function for $h>0$.

3.1 A One-Dimensional Example

To test the goodness of the proposed methodology we have carried out some simulations of model (2) and (3). In particular the signal $X(t)$ follows a stationary second order autoregressive model (AR(2), Box, Jenkins, 1970) with autoregressive parameters $\phi_1=0.5$, $\phi_2=-0.2$, and driving noise $\sigma^2=1$. Of course we do not observe $X(t)$ directly, but rather $Y(t)$ and $Z(t)$, which have been obtained by adding the internally and mutually Gaussian uncorrelated noises $\zeta(t)$ and $\eta(t)$ with variances $\sigma_\zeta^2=1$ and $\sigma_\eta^2=0.8$. Table 1 below, shows the results of estimation from 100 simulations of the two noise variances. In particular it focuses on bias, standard error (S.E.) and mean square error (MSE) for different lengths of the time series. It can be seen that the

estimator performs well and, as expected, the mean square error decreases as the number of observations increases.

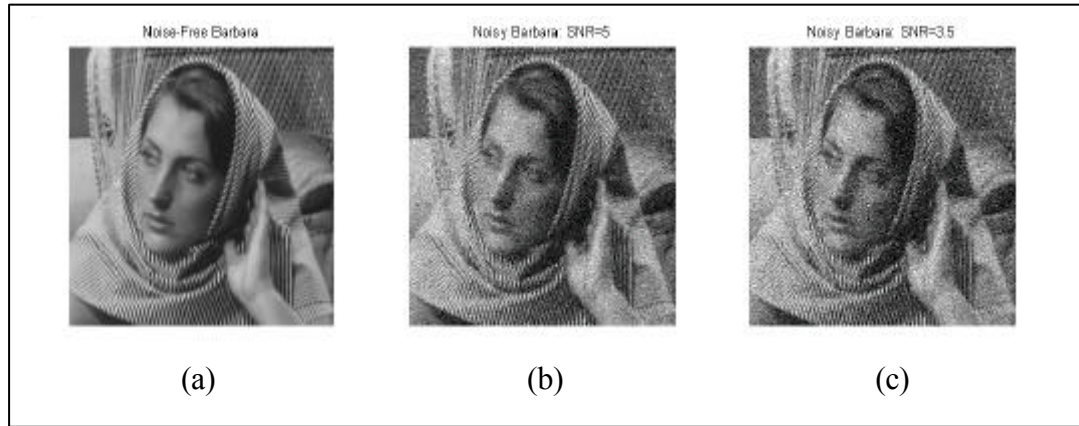
Table 1: Bias (Standard Errors) and MSE of the noise variances estimates

Number of Observations	Bias and S.E. $\hat{\sigma}_\zeta^2$	MSE $\hat{\sigma}_\zeta^2$	Bias and S.E. $\hat{\sigma}_\eta^2$	MSE $\hat{\sigma}_\eta^2$
50	- 0.018 (0.304)	0.093	0.016 (0.255)	0.065
100	0.007 (0.238)	0.057	0.006 (0.208)	0.043
150	- 0.013 (0.183)	0.034	- 0.002 (0.161)	0.026
200	- 0.001 (0.157)	0.025	0.013 (0.142)	0.020
250	0.003 (0.130)	0.017	0.006 (0.123)	0.015
300	- 0.007 (0.122)	0.015	0.003 (0.119)	0.014

3.2 A Two-Dimensional Example

As two-dimensional example we present the case of estimating the measurement noise variances of two noisy images which have the same “true image” in common. Figure 1(a) shows the noise-free 256×256 image of “Barbara”. Figure 1(b) and Figure 1(c) show the noisy images with different Signal-to-Noise-Ratios (SNR, i.e. the ratio between the signal and noise variances). Both figures are obtained by adding Gaussian uncorrelated noises $\zeta(i,j)$ and $\eta(i,j)$ with variances $\sigma_\zeta^2 = 570$ and $\sigma_\eta^2 = 900$.

Figure 1: Original and noisy Barbara image



By extending formulas (9) and (10) to the bidimensional case we obtained, for $\mathbf{h}=(0,0)$, the estimates $\hat{\sigma}_\zeta^2 = 568.729$ and $\hat{\sigma}_\eta^2 = 903.260$ which, considering the quite low values of the SNR, may be considered an excellent result.

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