

CUSUM Control Charts in the Presence of Correlation

Carte di Controllo CUSUM per Osservazioni Correlate

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Riassunto: Nel controllo statistico di processo si assume tradizionalmente che osservazioni consecutive nella traiettoria di un processo siano indipendenti. Spesso, tuttavia, nella pratica industriale, le osservazioni risultano essere serialmente correlate. In tal caso le carte di controllo hanno caratteristiche molto diverse rispetto a quelle nominali. In particolare, il numero medio di falsi allarmi può essere molto più elevato rispetto al valore prefissato. In questo articolo si concentrerà l'attenzione su un modello già analizzato in letteratura, per estendere la struttura di correlazione ad una classe più ampia e più flessibile. L'obiettivo è quello di valutare l'impatto di tale correlazione sulla carta CUSUM, in modo da adattare l'applicazione di tale carta in presenza di correlazione, affinché le caratteristiche effettive coincidano con quelle nominali.

Keywords: Statistical Process Control, Autoregressive Moving Average Model, Average Run Length.

1. Introduction

Traditional statistical process control assumes that consecutive observations from a process are independent. Often in industrial practice, however, in continuous as well as discrete processes, observations are actually serially correlated. See, for example, Harris and Ross (1991) for an extensive description of practical applications.

Under such conditions, traditional SPC procedures, designed for processes with independent observations, can have properties much different than expected when applied in monitoring a process with significant correlation. In particular, the average false alarm rate can be much higher than expected. A very high false alarm rate will cause process personnel to waste effort in unproductive searches for special causes, leading ultimately to a loss of confidence in the control chart.

This problem has been quite often studied in recent years, with reference to different models for the correlation structure and to the application of different control charts (for a review, see for example the introduction in Lu and Reynolds (1999)).

In this paper attention will focus on the model described, among others, in VanBrackle and Reynolds (1997), Lu and Reynolds (1999, 2001). The aim is to extend the correlation structure to a more general and more flexible class and evaluate the effect of this correlation structure on the CUSUM chart properties. In particular indications will be given to set the reference values of the CUSUM chart so that the effective false alarm rate is equal to the nominal one and a correct power comparison is possible.

2. The model

Suppose that samples of size $n \geq 1$ are taken from a process at times $t = 1, 2, \dots$ and let X_{ti} represent the i -th observation from the process at time t . Assume that X_{ti} can be written as

$$X_{ti} = \mu_t + \varepsilon_{ti}, \quad (1)$$

where ε_{ti} is a gaussian random error with zero mean and variance σ_ε^2 , μ_t is a random variable which represents the mean of the process at time t . With reference to that component, in papers previously mentioned it is assumed that $\{\mu_t\}$ is an AR(1) process with mean ξ :

$$\mu_t = (1 - \phi)\xi + \phi\mu_{t-1} + \alpha_t, \quad t = 1, 2, \dots \quad (|\phi| < 1); \quad (2)$$

here random shocks α_t are normally distributed with zero mean, variance σ_α^2 and independent from errors ε_s , $\forall (s, t)$. The distribution of the mean process $\{\mu_t\}$ is thus specified by (2) apart from the initial value μ_o . If we assume $\mu_o \sim \mathcal{N}(\xi, \sigma_\alpha^2/(1 - \phi^2))$, then μ_t will have that same distribution for $\forall t = 0, 1, 2, \dots$. This result together with equation (1) implies

$$X_{ti} \sim \mathcal{N}(\xi, \sigma_X^2), \quad \forall i = 1, 2, \dots, n, \quad t = 1, 2, \dots, \quad (3)$$

where

$$\sigma_X^2 = \sigma_\mu^2 + \sigma_\varepsilon^2 = \frac{\sigma_\alpha^2}{(1 - \phi^2)} + \sigma_\varepsilon^2. \quad (4)$$

A control chart for monitoring a process in order to detect changes in the overall mean $\xi = E(\mu_t)$ will be based, for $n \geq 1$, upon the sample mean observed at time t :

$$\bar{X}_t = \frac{1}{n} \sum_{i=1}^n X_{ti} \sim \mathcal{N}(\xi, \sigma_{\bar{X}}^2), \quad \text{with} \quad \sigma_{\bar{X}}^2 = \sigma_\mu^2 + \frac{\sigma_\varepsilon^2}{n}. \quad (5)$$

It is convenient to have an expression for the proportion of $\sigma_{\bar{X}}^2$ due to the term σ_μ^2 (i.e., for the proportion of variance in the process due to the correlated component):

$$\psi = \frac{\sigma_\mu^2}{\sigma_{\bar{X}}^2} = \frac{\sigma_\mu^2}{\sigma_\mu^2 + \frac{\sigma_\varepsilon^2}{n}}. \quad (6)$$

Observe that the effect of n is to increase the weight of σ_μ^2 on $\sigma_{\bar{X}}^2$; this is the reason why there is no loss of generality in considering the simplest case $n = 1$: different situations can be modeled through the parameter ψ .

Model (1) with (2) is called by Lu and Reynolds (1999) AR(1) *observed with random error*. The aim of this paper is to extend that model to an ARMA(1,1) *observed with random error*, i.e. we will assume, instead of (2),

$$\mu_t = (1 - \phi)\xi + \phi\mu_{t-1} + \alpha_t + \theta\alpha_{t-1}, \quad t = 1, 2, \dots \quad (|\phi|, |\theta| < 1). \quad (7)$$

Observe that Box et al. (1994) show that the AR(1) process observed with random error, (1) and (2), is equivalent to an ARMA(1,1) for X_{ti} . Here that assumption is extended, since the more flexible class ARMA(1,1) is used to model the mean process, μ_t , to which the random error ε_{ti} is further added.

In this case, if we assume

$$\mu_o \sim \mathcal{N}\left(\xi, \frac{1 + 2\phi\theta + \theta^2}{1 - \phi^2} \sigma_\alpha^2\right), \quad (8)$$

then μ_t has that same distribution $\forall t = 0, 1, 2, \dots$.

3. CUSUM control charts

It is well-known (see, e.g., VanDobben De Bruyn (1968)) that control charts based upon cumulative sums are more efficient than traditional Shewhart charts in detecting shifts in the process mean.

According to Page (1954), define the process S_t^+ , $t = 0, 1, 2, \dots$, as

$$\begin{aligned} S_0^+ &= 0 \\ S_t^+ &= \max\{0, S_{t-1}^+ + (X_t - k)\} \quad t = 1, 2, \dots \end{aligned} \quad (9)$$

and signal if $S_t^+ > h$. If we define in dual way the process S_t^- , $t = 0, 1, 2, \dots$, we obtain a two-sided control chart, which signals if either S_t^+ or S_t^- does. Parameters k and h are decided according to the desired average rate of false alarms and the power of the procedure in detecting a certain amount of mean shift from the target ξ_o (standardized as $d = \frac{\xi - \xi_o}{\sigma_X}$).

VanBrackle and Reynolds (1997) present the effect of correlation among successive X_t values (modeled as an AR(1) observed with error) on the performance of a CUSUM control chart in the following way: k and h are set to provide a desired in control Average Run Length (ARL(0)), namely the expected number of samples required to signal if the process is actually in control ($d = 0$); that value is calculated under the assumption of uncorrelated observations (i.e., $\phi = 0$ in (2)); given k and h , $ARL(d)$, $d \geq 0$, is calculated for different values of ϕ . That comparison is however misleading, since it well-known that ARL(0) decreases as ϕ increases: the power of the procedure for different ϕ values cannot be compared.

A more useful approach in order to provide specific indications for the control chart design is attained by setting k and ARL(0) to desired values and evaluate, *for different values of ϕ* , the alarm threshold, h , necessary to obtain them. In this way it is possible to obtain the correct false alarm rate for the control chart whenever an estimate for ϕ is available, or, at least, compare in a correct way the performance of the control chart for different correlation levels. This is the approach followed, for the simpler model (2), by Lu and Reynolds (2001), where some reference values are given and the comparison is correctly performed.

In Table 1 a more complete grid (although not exhaustive due to the limited amount of space) is presented for the more general model (7). Here $\psi = 0.9$, $k = 0.5$ and h was calculated so that $ARL(0) \simeq 370$ (this is the customary value arising in “three sigma” Shewhart charts). Each row of the table corresponds to a combination of ϕ and θ values (only combinations such that $ACF(1) = \text{Corr}(\mu_t, \mu_{t-1}) > 0$, i.e. $\phi + \theta > 0$, have been considered, since, in practice, this is the most relevant case).

4. Conclusions

From Table 1 it is evident that the power of the control chart decreases as the correlation increases. However, if $ACF(1) < 0.3$, the control chart detects an intermediate shift in the process mean ($d \geq 1$) quite fast. With respect to results in Lu and Reynolds (2001) it is interesting to observe that both θ and ϕ affect the performance of the chart: given ACF(1), different combinations of (ϕ, θ) produce a different time to detect an out-of-control state.

Table 1: Steady state ARL for CUSUM charts when $\psi = 0.9$, $k = 0.5$ for a selected number of combinations, (ϕ, θ) , $\phi + \theta > 0$; h is set to obtain $ARL(0) \simeq 370$.

ARL(d)				d							
ACF(1)	θ	ϕ	h	0.5	1.0	1.5	2.0	2.5	3.0	4.0	5.0
0.0000	0.0	0.0	4.775	33.87	9.29	5.06	3.56	2.77	2.30	1.81	1.46
0.1923	-0.2	0.0	5.98	41.02	11.54	6.26	4.34	3.35	2.74	2.10	1.78
0.2000	0.0	0.2	6.18	42.50	11.88	6.52	4.48	3.43	2.82	2.13	1.82
0.2091	0.2	0.4	6.24	43.92	12.26	6.59	4.53	3.47	2.84	2.15	1.83
0.2236	0.4	0.6	6.20	44.93	12.50	6.56	4.46	3.42	2.80	2.11	1.81
0.2600	0.6	0.8	6.04	47.71	12.87	6.43	4.36	3.31	2.71	2.05	1.76
0.3448	-0.4	0.0	7.24	48.46	13.99	7.51	5.15	3.95	3.23	2.41	2.00
0.3714	-0.2	0.2	7.74	50.31	14.82	8.17	5.51	4.20	3.43	2.55	2.08
0.4000	0.0	0.4	8.26	54.09	15.86	8.66	5.79	4.46	3.61	2.68	2.16
0.4400	0.2	0.6	9.00	59.76	17.74	9.45	6.30	4.77	3.91	2.88	2.30
0.4412	-0.6	0.0	8.36	53.32	15.80	8.71	5.90	4.53	3.67	2.73	2.21
0.4878	-0.8	0.0	9.40	58.92	17.78	9.78	6.60	5.05	4.09	3.03	2.46
0.4909	-0.4	0.2	9.26	59.52	17.54	9.52	6.53	4.96	4.03	2.98	2.39
0.5231	0.4	0.8	10.70	74.97	22.37	11.42	7.44	5.53	4.47	3.27	2.64
0.5400	-0.2	0.4	10.22	63.38	19.59	10.54	7.15	5.42	4.38	3.22	2.59
0.5600	-0.6	0.2	10.62	64.35	19.74	10.89	7.36	5.60	4.54	3.35	2.70
0.5919	-0.8	0.2	11.86	69.90	22.38	12.12	8.28	6.27	5.11	3.72	2.98
0.6000	0.0	0.6	11.83	72.23	23.05	12.41	8.24	6.17	4.99	3.64	2.90
0.6270	-0.4	0.4	12.14	72.67	23.06	12.47	8.43	6.37	5.13	3.74	3.01
0.6739	-0.6	0.4	13.84	80.19	25.96	14.18	9.53	7.28	5.85	4.24	3.37
0.6948	-0.8	0.4	15.40	85.32	28.62	15.76	10.57	8.04	6.45	4.70	3.72
0.7000	-0.2	0.6	14.40	84.00	27.28	14.86	9.93	7.47	5.98	4.34	3.45
0.7000	0.2	0.8	15.42	95.15	31.05	16.25	10.69	7.88	6.25	4.51	3.57
0.7561	-0.4	0.6	17.00	95.82	32.20	17.36	11.88	8.75	7.04	5.08	4.03
0.7847	-0.6	0.6	19.24	103.08	35.65	19.77	13.25	9.99	7.95	5.71	4.51
0.7969	-0.8	0.6	21.44	110.96	39.86	21.84	14.82	11.12	8.85	6.37	5.02
0.8000	0.0	0.8	19.8	111.81	39.12	20.75	13.76	10.07	7.94	5.70	4.50
0.8529	-0.2	0.8	23.70	122.38	45.65	24.68	16.33	12.18	9.55	6.81	5.34
0.8800	-0.4	0.8	27.80	140.12	52.15	29.02	19.24	14.31	11.30	8.02	6.26
0.8931	-0.6	0.8	31.22	147.17	57.51	32.50	21.64	16.16	12.79	9.01	7.04
0.8987	-0.8	0.8	34.82	155.64	63.75	35.97	24.09	17.97	14.31	10.09	7.84

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