

Non-symmetric Correspondence Analysis for Ordered Two-way Contingency Table¹

L'Analisi Non-Simmetrica delle Corrispondenze per una tabella di contingenza ordinale

Luigi D'Ambra

Department of Mathematics and Statistics -University of Naples
dambra@unina.it

Rosaria Lombardo

Department of Management, Accounting and Quantitative Methods S U N
rosaria.lombardo@unina2.it

Pietro Amenta

Department of Economics, Mathematics and Statistics-University of Lecce
amenta@unile.it

Riassunto: La ricerca di variabili latenti in una tabella di contingenza a due vie in presenza di una variabile ordinale non può essere affrontata con le metodologie classiche dell'analisi multidimensionale dei dati come l'Analisi delle Corrispondenze (Greenacre 1984) o l'Analisi Non Simmetrica delle Corrispondenze (D'Ambra & Lauro 1989), poiché tali metodologie non tengono conto di una struttura ordinale nelle modalità riga e/o colonna né di un'eventuale trend presente nei dati. In questo articolo si propone un nuovo approccio per lo studio della dipendenza tra una variabile ordinale e un'altra non ordinale utilizzando i polinomi ortogonali (Emerson 1968) al fine di evidenziare il trend dei dati.

Keywords: Ordinal Contingency Table, Non-Symmetric Correspondence Analysis, Orthogonal Polynomials.

1. Introduction

Pearson's phi-square as tau numerator (Goodman & Kruskal 1954, Light & Margolin 1971), which are on the basis of Correspondence Analysis (CA) and Non-symmetric Correspondence Analysis (NSCA), respectively, are important statistics for the analysis of categorical data, but they do not take the ordering of the row and column categories into account. Sometimes plotting the categories of ordered variables it will be possible to observe a trend which does not play any role in our calculated statistics. What we want is a statistic which takes order into account.

In literature a great variety of works deal with the information about the structure of the ordinal variable. We can distinguish between methods which impose the order

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constraint on: 1) the criterion function (Gifi 1990, Ramsay 1988, Tibshirani *et al.* 1997), or on 2) the estimated score vector (Nishisato 1980, Nishisato *et al.* 1975, D'Ambra *et al.* 2000), either 3) adding external informations (Bockenholt, Bockenholt 1990, Amenta, D'Ambra 1996) by means of an orthogonal contrast matrix, which represents, in special case, the coefficients of orthogonal polynomials. We may ask how sensitive are ordinal analysis to the choice of scores. There is no simple answers, in general different score systems give quite different results (Agresti 1990).

It is clear that the imposition of additional constraints on the criterion function or on the score vectors can lead to a solution which is less optimal than the unconstrained one. This loss should be minimal while the gain in the interpretative value of an ordered solution should be larger.

In this paper we make the ordinal information as part of our analysis, and differently from the previous analysis, the computed solution does not imply any loss in the criterion function. We treat the problem of dependence in a contingency table, when a variable is ordered and logically antecedent with respect to the other one, by using the orthogonal polynomials of Emerson (1968).

The advantage of using orthogonal polynomials relies in the fact that the order information is considered in the analysis, and the resulting scoring scheme permits a clear interpretation of the linear, quadratic or higher order trend components.

The following approach will allow to identify important characteristics in the behavior of the non-ordered variable across the categories of the ordered variable.

From the representation it will be possible to show the categories within a variable, pointing out how these "within variable categories" may or may not be different.

The criterion function represents the numerator of a known measure of predictability, the τ (D'Ambra & Lauro, 1989). This new methodology, called Singly Ordered Non Symmetric Correspondence Analysis (*SONSCA*), is based on the numerator decomposition of the asymmetric association measure τ by the Emerson's orthogonal polynomials, similarly as proposed for the Pearson chi-squared statistic (Best & Rayner 1996, Beh 1997, 1999).

2. Orthogonal Polynomials for ordered contingency tables

In this section a brief description of the orthogonal polynomials presented by Emerson (1968) for contingency tables is given.

Let $\mathbf{N}$ and $\mathbf{P}=\mathbf{N}/n=\{n_{ij}/n=p_{ij}\}$ be the absolute and relative two-way contingency tables, respectively, of dimension I,J , with I the number of categories of the response variable and J the category number of the predictor variable, observed on the same n individual. Let $p_{i.}=\sum_{j=1}^J p_{ij}$ and $p_{.j}=\sum_{i=1}^I p_{ij}$ be the row and column marginals,

respectively. Suppose that $\mathbf{P}$ has ordered column categories, the set of column orthogonal polynomials, computed by the general recurrence formula (Emerson 1968, Beh 1997), are placed into a (J,J) matrix $\mathbf{B}^*$ whose general term is $b_v^*(j)$. The column polynomials are orthogonal with respect to the diagonal metric $\mathbf{D}_j=\{p_{.j}\}$:

$$\sum_{j=1}^J p_{.j} b_{j(v)}^* b_{j(v')}^* = \begin{cases} 1 & \text{if } v = v' \\ 0 & \text{if } v \neq v' \end{cases} \quad (2.1)$$

Observe that the transpose matrix of orthogonal polynomials is equal to the inverse $\mathbf{B}^{*'} = \mathbf{B}^{*-1}$ and defining $\hat{\mathbf{B}} = \mathbf{B}^* \mathbf{D}_j^{1/2}$ the orthogonal constraint can be written as $\hat{\mathbf{B}}' \hat{\mathbf{B}} = \hat{\mathbf{B}} \hat{\mathbf{B}}' = \mathbf{I}$.

Assuming that the first column of $\mathbf{B}^* = \{\mathbf{b}_{j(0)}^*\}$ is the trivial orthogonal polynomials whose elements are all equal to 1, and also that $\mathbf{b}_{j(-1)}^* = 0$ (for $j=1, \dots, J$). The general recurrence relation of Emerson for computing the column orthogonal polynomials, can be written as

$$b_{j(v)}^* = S_v \left[(s(j) - T_v) b_{j(v-1)}^* - V_v b_{j(v-2)}^* \right] \quad (2.2)$$

where

$$T_v = \sum_{j=1}^J p_{.j} s(j) b_{j(v-1)}^{*2} \quad (2.3)$$

$$V_v = \sum_{j=1}^J p_{.j} s(j) b_{j(v-1)}^* b_{j(v-2)}^* \quad (2.4)$$

and

$$S_v = \left\{ \sum_{j=1}^J p_{.j} s(j)^2 b_{j(v-1)}^{*2} - T_v^2 - V_v^2 \right\}^{-1/2} \quad (2.5)$$

The general recurrence relations of (2.2)-(2.5) are only able to be used when a set of ordered column categories $s(j)$ (for $j=1, 2, \dots, J$) is used. Observe that when the row categories are assumed to be ordered, a similar set of row orthogonal polynomials, $a_{i(u)}^*$ for ($u=1, 2, \dots, I$) can be derived. The orthogonal polynomials are not new and have recently been used in ordered *CA* (Best & Rayner 1996). These polynomials could be computed with respect to different scoring schemes defining the ordered structure of the categories such as natural scores, midrank scores (Nair 1986), Nishisato's scores (Nishisato & Arri 1975) or regression ones (D'Ambra *et al.* 2000), different schemes can have distinct affects on the analysis (Agresti 1990). The general scheme that will be used in this work is the natural score scheme expressed by the integer values $\{j\}$ for $j=1, \dots, J$.

3. Non-Symmetric Correspondence Analysis for ordered categories

In this section after briefly resume *NSCA* (D'Ambra & Lauro 1989), we discuss Singly Ordered *NSCA* (*SONSCA*), in two different versions, to analyse the dependence between a categorical response and an ordinal predictor variable.

Consider the row (column) marginals, as the estimated probability that an individual is in row i (column j) irrespective of the column (row) who belongs to.

Define $\Pi = (\pi_{ij})$ as the centered column profile matrix whose general term is

$$\pi_{ij} = \frac{p_{ij}}{p_{.j}} - p_{i.} \quad \text{with} \quad \sum_i \pi_{ij} = \sum_j p_{.j} \pi_{ij} = 0$$

In order to visualize the dependence structure between the criterion variable (rows) and the exploratory variable (columns) onto a suitable subspace R^I , *NSCA* looks for an ortho-normal basis which allows to take into account a large part of the inertia of point cloud $\Pi = (\pi_{ij})$, i.e. the numerator of the tau index, so as to give the most representative image of multidimensional data. The tau numerator represents the absolute increase in predictability of a response category given the knowledge of the predictor one.

$$N_{\tau} = \|\Pi\|^2 = \sum_{i=1}^I \sum_{j=1}^J p_{.j} \pi_{ij}^2 = \sum_{i=1}^I \sum_{j=1}^J p_{.j} \left(\frac{p_{ij}}{p_{.j}} - p_{i.} \right)^2 \quad (3.1)$$

By a generalisation of the Singular Value Decomposition, the full rank ($J < I$ and $M=J$) representation of the columns and rows of Π is found:

$$\pi_{ij} = \sum_{u=1}^M \lambda_u a_{iu} b_{ju} \quad (3.2)$$

where λ_u is the singular value and the $\mathbf{a}_u = \{a_{iu}\}$ and $\mathbf{b}_u = \{b_{ju}\}$ are the orthonormal singular vectors in an unweighted and weighted metric, respectively

$$\sum_{i=1}^I a_{iu} a_{iu'} = \begin{cases} 1 & \text{if } u = u' \\ 0 & \text{if } u \neq u' \end{cases}$$

$$\sum_{j=1}^J p_{.j} b_{ju} b_{ju'} = \begin{cases} 1 & \text{if } u = u' \\ 0 & \text{if } u \neq u' \end{cases}$$

3.1 An Alternative Approach to *NSCA*: Singly Ordered *NSCA*

Various papers dealt with partitioning the symmetric chi-squared statistic with one or two ordered categories (Best & Rayner 1996). In this paper we propose the partitioning of the numerator of tau statistic, through the use of orthogonal polynomials.

The orthogonal polynomials permits to take into account the ordered structure in the τ optimization, the technique proposed is called Singly Ordered *NSCA* (*SONSCAI*).

This approach enables to identify what effect the ordered columns has upon the rows.

Using orthogonal polynomials we are able to partition the inertia in linear, quadratic and higher order components, taking into account the order information in the dependence study between variables. Suppose we have a two-way contingency table with ordered columns and unordered rows. We state that the tau statistic numerator can be partitioned with respect to the orthogonal polynomials so that:

$$N_{\tau} = \|\mathbf{Z}\|_{\mathbf{D}_j}^2 = \sum_{i=1}^I \sum_{v=0}^{J-1} z_{iv}^2 \quad \text{where} \quad z_{iv} = \sum_{j=1}^J \left(\frac{p_{ij}}{p_{.j}} - p_{i.} \right) \hat{b}_{j(v)} \quad (3.3)$$

note that $\hat{b}_{j(v)}$ is the generic element of the orthogonal polynomials of order v associated to the j^{th} column category. The advantage of this transformation is to preserve all the information contained in the data, pointing out the trend given by the ordinal variable.

In matrix notation, we can write $\mathbf{Z} = \mathbf{\Pi} \hat{\mathbf{B}}$ where $\hat{\mathbf{B}} = \{\hat{b}_{j(v)}\}$ is the $J \times J$ matrix containing the J orthogonal polynomials. To prove (3.3), we consider the orthonormalization constraint on the orthogonal polynomials (formula 2.1) and yield

$$N_{\tau} = \sum_{i=1}^I \sum_{j=1}^J \sum_{v=0}^{J-1} \left(\frac{p_{ij}}{p_{.j}} - p_{i.} \right)^2 p_{.j} \hat{b}_{j(v)}^2 = \sum_{i=1}^I \sum_{j=1}^J p_{.j} \mathbf{\Pi}_{ij}^2$$

proving equation (3.3). Therefore we can write

$$N_{\tau} = \sum_{i=1}^I \sum_{v=0}^{J-1} (z_{iv}^2 + z_{i0}^2) = \sum_{i=1}^I \sum_{v=1}^{J-1} z_{iv}^2 = \sum_{v=1}^{J-1} N_{\tau v}$$

$$\text{because } \sum_{i=1}^I z_{i0} = \sum_{i=1}^I \sum_{j=1}^J \left(\frac{p_{ij}}{p_{.j}} - p_{i.} \right) \hat{b}_{j(0)} = \sum_{i=1}^I \sum_{j=1}^J \left(\frac{p_{ij}}{p_{.j}} - p_{i.} \right) = 0$$

It can be shown that if assume that $z_{iv} \approx N(0,1)$ then $U^2 = (N-1)(I-1)\tau$ is under the null hypothesis of independence distributed as a $\chi_{(I-1)(J-1)}^2$ (Agresti 1990, Kroonenberg & Lombardo 1999).

SONSCA1 allows to represent the column categories by means of the coordinates given

by $g_{ji}^* = \sum_{v=1}^{J-1} \hat{b}_{j(v)} z_{iv}$ with $\sum_j p_{.j} g_{ji}^{*2} = \sum_{v=1}^{J-1} z_{iv}^2$. Nevertheless this approach does not permit

to represent the row categories by latent variables or the dependence between the column and row categories. For this reason we propose an alternative approach to singly ordered *NSCA* that we call *SONSCA2*.

3.2 An integrated Approach using singular vectors and orthogonal polynomials

Suppose we have a two-way contingency table with ordered columns and non-ordered rows. Then we can partition the tau's numerator using orthogonal polynomials for the ordered columns (equation 2.2) and singular vectors (equation 3.2) for the unordered rows.

The *SONSCA2* of the N_{τ} partition, using the orthogonal polynomials, determines the latent variables in terms of singular vectors and trend components of the row and column categories, respectively. It will be possible to visualize data trend, and measure the proportional increase in predictability for a row category given a column one.

Furthermore the numerator of the τ statistics, i.e. the inertia of points, can be partitioned via bivariate moments for two-way contingency table. For example these moments are expressed in terms of the location component, that is the linear measure of the difference in the mean value of a variable, either of the dispersion component, that is a quadratic measure of the spread of the categories within a variable, or higher order components. Let be $\mathbf{Z} = \left\{ z_{u(v)} = \sum_{i=1}^I \sum_{j=1}^J \pi_{ij} a_{iu} \hat{b}_{j(v)} \right\}$, where $\mathbf{A} = \{a_{iu}\}$ is the matrix (I, M) of the left singular vectors, while $\hat{\mathbf{B}} = \{\hat{b}_{j(v)}\}$ is the matrix (J, J) of the J column orthogonal polynomials. The general term $z_{u(v)}$ means that each principal axis from *NSCA* can be partitioned into column component values, so that it will be possible to determine the dominant source of variation of the ordered columns. In matrix notation $\mathbf{Z}$, which allows the decomposition of the centered column profile in terms of singular vectors and orthogonal polynomials, can be written as

$$\mathbf{Z} = \mathbf{A}' \mathbf{\Pi} \mathbf{B}^* \quad (3.6)$$

with $\mathbf{A}'\mathbf{A} = \mathbf{A}\mathbf{A}' = \mathbf{I}$ and $\hat{\mathbf{B}}'\hat{\mathbf{B}} = \hat{\mathbf{B}}\hat{\mathbf{B}}' = \mathbf{I}$, so that the weighted centered column profile is equal to $\mathbf{\Pi} = \mathbf{AZ}\hat{\mathbf{B}}'$ whose general term is

$$\pi_{ij} = \sum_{u=1}^M \sum_{v=0}^{J-1} a_{iu} z_{(u)v} \hat{b}_{j(v)}$$

Eliminating the trivial solution the previous formula yields to

$$\pi_{ij} = \sum_{u=1}^M \sum_{v=1}^{J-1} a_{iu} z_{(u)v} \hat{b}_{j(v)}$$

It can easily be shown that the tau numerator of the singly ordered analysis, using singular vectors for the ordered categories and orthogonal polynomials for the non ordered one, can be decomposed so that

$$N_{tau} = \sum_{i=1}^I \sum_{j=1}^J p_{.j} \pi_{ij}^2 = \sum_{u=1}^M \sum_{v=1}^J z_{u(v)}^2 \quad (3.7)$$

In general the v^{th} column profile component can be calculate by $N_v = \sum_{u=1}^M z_{u(v)}^2$.

An interesting property of formula (3.7) is that the total inertia can be partitioned into bivariate moments (see section 4).

3.3 The co-ordinate system in *SONSCA2*.

In order to display the dependence relationship between the row and column categories, the classical plotting system requires to compute the product between singular-vectors and singular-values (Greenacre 1984). In this paper we will use a combination of the classical approach (left singular vectors) for the non-ordered set of categories, and the ordinal approach (orthogonal polynomials) for the ordered categories. The relationship

between the row and column categories is based on the $\mathbf{Z}$ matrix, while the classical approach considers the singular values. Therefore for *SONSCA2* the row and column coordinates may be written as:

$$f_{iv} = \sum_{u=1}^M a_{iu} z_{u(v)}$$

$$g_{ju}^* = \sum_{v=1}^{J-1} \hat{b}_{j(v)} z_{u(v)}$$

In matrix notation we can write

$$\mathbf{F} = \mathbf{AZ}$$

$$\mathbf{G}^* = \hat{\mathbf{B}}\mathbf{Z}'$$

Substituting the expression of $\mathbf{Z}$ given by formula 3.6 and using the orthogonal polynomial constraint, we yield the alternative expressions for the row and column coordinates, respectively.

$$\mathbf{F} = \Pi \hat{\mathbf{B}} \quad \mathbf{G}^* = \Pi' \mathbf{A}$$

furthermore it can be proved the following transition formulas

$$\mathbf{FZ}' = \mathbf{AA}' \Pi \hat{\mathbf{B}} \hat{\mathbf{B}}' \mathbf{D}_j \mathbf{G}^* = \Pi \mathbf{G}^*$$

$$\mathbf{G}^* \mathbf{Z} = \Pi' \mathbf{F}$$

Finally we observe that the column coordinate set results centered with respect to the origin $\sum_{j=1}^J p_{.j} g_{ju}^* = 0$ and the squared coordinate sets reconstruct the tau numerator (total inertia)

$$trace(\mathbf{F}'\mathbf{F}) = trace(\mathbf{G}^*{}'\mathbf{D}_j\mathbf{G}^*) = N_{tau}$$

When contingency tables have ordered columns and non-ordered row categories, then the $(i,j)^{th}$ cell probability can be calculated by the saturated reconstitution formula

$$p_{ij} = p_{i.} p_{.j} + p_{.j} \sum_{u=1}^M \sum_{v=1}^{J-1} a_{iu} z_{u(v)} \hat{b}_{j(v)}$$

3.4 Distances

In a geometric context one of the aim of *SONSCA2* is to portray the difference between the centred column profile in the column space, where it will be possible to evaluate the differences between column categories on the linear, quadratic or higher order component. The squared distance $d_J^2(j, j')$ between column profiles j and j' is

$$d_J^2(j, j') = \sum_{u=1}^M (g_{ju}^* - g_{j'u}^*)^2 = \sum_{u=1}^M \sum_{v=1}^{J-1} (\hat{b}_{j(v)} z_{u(v)} - \hat{b}_{j'(v)} z_{u(v)})^2$$

Furthermore the squared distance $d_I^2(i, i')$ between row categories is given by

$$d_I^2(i, i') = \sum_{v=1}^{J-1} \sum_{j=1}^J \frac{\hat{b}_{j(v)}^2}{p_{.j}} (p_{ij} - p_{i'j})^2 - (p_{i.} - p_{i'.})^2$$

4. Properties and interpretation of the trend components

The tau index numerator is partitioned into bivariate moments calculated using orthogonal polynomials for the ordered variable and generalised singular vectors for the non ordered one.

In this section we describe and expand upon the main properties which show the relationship between the singular values and the bivariate moments.

- **Property 1** The $\text{trace}(\mathbf{ZD}_J\mathbf{Z}') = \sum_{u=1}^M \sum_{v=1}^{J-1} z_{u(v)}^2 = \text{trace}(\Lambda^2)$, where Λ^2 is the diagonal eigenvalue matrix, is equivalent to the total inertia N_{tau} .
- **Property 2** The row component associated with the u^{th} principal axis is just the value of the u^{th} largest eigenvalue.

To show this, recall that the total inertia may be written in terms of bivariate moments or as eigenvalues, such that

$$N_\tau = \sum_{u=1}^M \sum_{v=1}^{J-1} z_{u(v)}^2 = \sum_{u=1}^M \lambda_u^2$$

so that

$$\lambda_u^2 = \sum_{v=1}^{J-1} z_{u(v)}^2 \quad \text{for } (u=1 \dots M)$$

Furthermore it can be shown that the column component on the v^{th} orthogonal polynomial is given by:

$$\sum_{u=1}^M z_{u(v)}^2 = \sum_{i=1}^I (f_{iv})^2$$

Consider the case when $v=1$, $N_{\tau \text{Linear}} = \sum_{u=1}^M z_{u1}^2$ is then called the *column location component* of the tau statistic numerator and describes the overall deviation of the centered column profile means. Higher order components can be calculated by

considering higher order of v . For example when $v=2$, $N_{\text{Dispersion}} = \sum_{u=1}^M z_{u2}^2$ is the *column dispersion component* and describes the difference in the centered column profiles in terms of the difference in their spread.

- **Property 3** *The row component values are arranged in descending order.*

This property follows directly from Property 2, as the eigenvalues are arranged in descending order.

- **Property 4** Note that the element $z_{1(i)}$ (linear component) represents the inner product between the ordered score categories (columns) and the non ordered ones (rows), weighted with column profile

$$z_{1(i)} = \sum_i \sum_j a_{i1} \pi_{ij} \hat{b}_{j(1)} = \sum_i \sum_j a_{i1} (j - \mu_j) \frac{p_{ij}}{p_{.j}} / \sigma_j$$

$$\text{with } \sigma_j = \sum_j p_{.j} j^2 - \left[\sum_j p_{.j} j \right]^2 \text{ and } \mu_j = \sum_j p_{.j} j$$

- **Property 5** *It is possible to identify which trend component (linear quadratic or higher order) contributes the most to a particular eigenvalue and hence principal axis.*

From Property 2, when the principal inertia of the u^{th} principal axis is the sum of squares of the bivariate moments. The eigenvalues may be interpreted by considering the linear-by-linear, linear-by-quadratic and higher order bivariate moments. Considering the largest eigenvalue we can write

$$\lambda_1^2 = z_{1(1)}^2 + z_{1(2)}^2 + \dots + z_{1(J-1)}^2$$

Property 5 allows us to isolate important bivariate moments for each principal axis which describe the relationship between the row and column categories. It will be possible to focalize on the contribution of the linear, quadratic, or higher order component on the u^{th} eigenvalue, which expresses the dependence relationship between the row and column categories. So this method allows for a decomposition of the singular values into location, dispersion and higher order components. The v^{th} orthogonal polynomial allows to dictate the trend of the column categories. It is clear that the axes of *SONSCA2* plot are constructed so that the first axis is the most important, the second axis is the second most important and so on.

Hence one of the major problems in order to analyse the dependence structure of mixed variables can be overcome by considering such methodology.

5. Conclusion

In this paper we have discussed new developments of *NSCA* for the application to two-way contingency tables with at least one ordered variable. In sociological, health or customer satisfaction studies where variables are often rated according to a Likert or

Thurstone scale, the ordinal structure of these variables needs to be considered. We have shown that the classical approach and the new one using orthogonal polynomials are mathematically equivalent, but the advantage of using orthogonal polynomials is that important association (linear by linear, linear by quadratic etc.) can be easily identified by observing the matrix of association $\mathbf{Z}$. The generalization to three-way contingency tables is in the perspectives of this work and practical examples comparing the different visual displays of the categories will be provided.

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