

The Long Range Dependence Paradigm for Finance

Il concetto di lunga memoria nell'ambito della teoria finanziaria

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Riassunto: Il paradigma della Long Range Dependence (LRD) riveste una grande importanza per la descrizione di dati di serie storiche generati da processi economici. La principale ragione è che la LRD caratterizza in modo naturale serie caratterizzate da una forte persistenza. Modelli lineari con LRD sono stati molto usati in macroeconomia. L'evidenza empirica basata su dati finanziari, d'altra parte, ha portato allo sviluppo di modelli non lineari con LRD, e in particolare modelli con volatilità stocastica. In questo lavoro si discutono sia l'evidenza empirica a supporto della LRD, sia gli aspetti teorici dei modelli introdotti sulla base di tale evidenza empirica.

Keywords: Self-similarity, time series, FARIMA, nonlinear time series, ARCH, stochastic volatility, arbitrage.

1. Introduction

Granger (1966) first pointed out that nonparametrically estimated power spectra of many economic variables, such as industrial production and commodity price indexes, suggested overwhelming importance of the low frequency components. Mandelbrot (1971) observed a self-similar behaviour in the distribution of speculative prices, and proposed continuous and discrete time fractional models, such as fractional Brownian motion; see Mandelbrot and Ness (1968). He then developed the Hurst rescaled range (hereafter R/S) analysis, originally introduced in Mandelbrot and Wallis (1969), for social measurement purposes. However, initial sizeable empirical success of the long-range dependence (LRD) concept in economics is certainly related to the autoregressive fractionally integrated moving average model (ARFIMA), proposed simultaneously by Hosking (1981) and Granger and Joyeux (1980), and the increased flexibility it brings to the Box-Jenkins linear time series methodology, in particular for modelling macroeconomic time series since Diebold and Rudebusch (1989). The relative success of the LRD concept in economics may also be attributed to the development of a rationale for its presence in macro-level economic and financial systems based on the aggregation of micro units. This was originally proposed in Robinson (1978) and further developed in Granger (1980), both in the context of contemporaneous aggregation of heterogeneous AR(1) micro-units.

Estimation of fully parametric long memory models such as ARFIMA appeared cumbersome, especially in the time domain, and, moreover, practical estimation of these models indicated that the estimated values of the long memory parameter was largely affected by the parametric specification of the short run dynamics part of the model, i.e. the ARMA component. Hence, an important reason which gave further impetus to the importance of LRD in economics, was the introduction by Geweke and Porter-Hudak (1983) of a semi-parametric estimator (hereafter GPH) for the estimation of the degree of LRD in a time series. In fact, GPH is robust to many forms of complicated short run dynamics.

Empirical finance research had a tremendous impact in emphasizing the importance of the LRD paradigm, both empirically and theoretically. On the one hand, the availability of very large time series of high-frequency data, allowed easier and more convincing detection of long memory. On the other hand, some empirical findings prompted the development of nonlinear time series models apt to fit the empirical distribution of asset returns, synthesized in a number of well known stylized facts, including slow decay of sample autocorrelations of squared returns. This has stimulated research aiming at establishing asymptotic theory for such models and deriving their implications in terms of asset pricing and risk management in arbitrage-free theoretical frameworks. These issues are the focus of section 2.

2. Long memory implications in finance

2.1 Long memory in asset prices.

Following the mixed empirical success of the LRD paradigm in macroeconomics, a great deal of research has focused on possible evidence of LRD in financial asset returns defined as $x_t = \log(P_t/P_{t-1})$, where P_t is the price of the asset. Mandelbrot (1971) first suggested the possibility that asset prices could exhibit LRD. Using the classical R/S analysis, Greene and Fielitz (1977) uncovered significant statistical evidence of LRD in daily returns on securities listed at the New York Stock Exchange. These results were challenged by Lo (1991) who proposed a modified R/S statistic, robustifying for possible additional short memory dynamics by taking into account the first q lags of the autocovariance of the observables. He concluded that there is no such clear evidence of LRD in the levels of asset returns.

Hence, the empirical evidence concerning LRD in the levels of financial asset returns is far from being clear-cut. Note that most of modern finance theory is based on the martingale difference assumption for the levels of asset returns, in turn a by-product of a martingale assumption for log-prices and therefore the simple presence of autocorrelated returns data in levels, however weak, poses considerable problems. Indeed, there is some empirical evidence of significant autocorrelation at the first few lags often imputed to non-synchronous trading and other market microstructure effects; see e.g. Andersen and Bollerslev (1997) for forex returns and Ding *et al.* (1993) for the Standard & Poor 500 index (*S&P 500*) return.

There is a general agreement on the fact that asset returns exhibit more memory in the squares than in the levels. This stylized fact was refined to the presence of LRD in many nonlinear transformations of absolute returns (including squared and log-squared returns). Using daily data of the *S&P 500* closing price index, Ding *et al.* (1993) show that there is significant autocorrelation for lags up to 10 years (approximately 2,500 lags with daily data) when considering the squares return x_t^2 and more generally power transformations such as $|x_t|^\alpha$ for various positive values of α . Considering a suitable $\alpha < 2$, the ACF of $|x_t|^\alpha$ would be well defined even when the x_t have unbounded fourth moment. This pattern is consistent with theoretical autocovariances of the x_t^2 decaying hyperbolically.

This finding has been corroborated since the availability high frequency tick-by-tick data. Henry and Payne (1997), Henry and Payne (1998), Andersen and Bollerslev (1997), Andersen *et al.* (1999) consider the returns from using the foreign exchange spot rates DM/\$ and Yen/\$ and, using semiparametric estimators such as the GPH, the local Whittle

and the average periodogram estimator of Robinson (1994), found evidence of covariance stationary LRD in the squared, log-squared and absolute returns. Henry and Payne (1997) and Henry and Payne (1998) use the optimal bandwidth selection procedures discussed in Henry (2001)

Since Tauchen and Pitts (1983), it has been thought that the existence of a stable relationship between stock returns and trading volumes could be exploited in forecasting speculative prices dynamics. Using daily data for the thirty stocks which contribute to the Dow Jones industrial average index, Lobato and Velasco (1999) provides empirical evidence of long memory in stock return volatility and trading volume, with the same degree of memory, although the hypothesis of fractional cointegration was rejected. A semiparametric estimator of the coherency was used.

Unlike macroeconomic time series, financial data and particularly high-frequency data make it very natural to use semi and nonparametric estimators, as the usual efficiency loss is compensated by the possibility of using very long spans of data. This, in part, explains why fully parametric long memory models have not been advocated in order to detect long memory in asset returns, using in turn the levels and the squares as observables. However, once the general statistical properties of the data are uncovered, precise and efficient estimation of the volatility of asset returns is crucial for evaluation of derivative securities linked to such assets (such as options and futures). Furthermore, accurate forecasting of volatility of financial assets is a key ingredient for risk management. Nonparametric probability density estimators for long memory variates (see Robinson (1991a)) do extend to estimation of conditional moments but only for a finite number of lagged dependent variables. On the other hand, semiparametric long memory estimators, such as Robinson (1995b) and Robinson (1995a), focus precisely on long-horizon past information but ignore short run dynamics. Therefore, in order to model the intrinsically non-Markovian dynamics of LRD in conditional variance, and for practical pricing of financial assets, fully parametric long memory models of changing volatility must be considered.

2.2 Long memory volatility models.

Financial asset returns display a number of regularities across assets and periods, in particular dynamic conditional heteroskedasticity. This clearly cannot be accounted for by linear models. Furthermore, as originally pointed out by Black (1976), the data exhibit non-Gaussian kurtosis and dynamic asymmetries such as leverage effect, expressed by a negative cross-correlation between current returns (x_t) and future squared returns (x_{t+u}^2 with $u > 0$). The ARCH model of Engle (1982) represents the most famous nonlinear time series model apt to account for such features, except for dynamic asymmetries. Denoting by σ_t^2 the conditional variance $\text{var}(x_t^2 \mid x_{t-1}, x_{t-2}, \dots)$, the ARCH(p) postulates that

$$\sigma_t^2 = \omega + \alpha_1 x_{t-1}^2 + \dots + \alpha_p x_{t-p}^2 \quad a.s., \quad (1)$$

where $\omega > 0$, $\alpha_i > 0$ ($i = 1, \dots, p$). Therefore, the conditional variance is a symmetric function of past observations. A plethora of extensions have been developed in order to take other features into account, in particular dynamic asymmetries; see Bollerslev *et al.* (1994) for a recent survey.

Robinson (1991b) generalized (1) to

$$\sigma_t^2 = \tau^2 + \sum_{j=1}^{\infty} \psi_j (x_{t-j}^2 - \tau^2) \quad a.s., \quad \sum_{i=1}^{\infty} \psi_i < \infty, \quad (2)$$

with $\tau^2 > 0$ and $\psi_j \geq 0$. This includes (1) when $\psi_j = 0$ for all $j > p$. Robinson (1991b) introduced the ARCH(∞) model (2) and

$$\sigma_t^2 = \left(\omega + \sum_{j=1}^{\infty} \phi_j x_{t-j} \right)^2 \quad a.s., \quad \sum_{i=1}^{\infty} \phi_i^2 < \infty, \quad (3)$$

as alternative hypotheses in deriving score tests of no-ARCH with optimal efficiency against such alternatives. Note that for (3) there is no need to impose nonnegativity on ω and on the ϕ_i . Model (3), called linear ARCH (LARCH), was further considered in Giraitis *et al.* (1998) who characterize the low-order statistical properties of the model. In particular, they establish sufficient conditions for LRD in the x_t^2 and weak convergence of their partial sums to fractional Brownian motion $B_H(t)$.

In order to impose martingale difference levels and autocorrelated squares, the data generating process for x_t is routinely defined by

$$x_t = \epsilon_t \sigma_t, \quad (4)$$

where the so-called rescaled innovations ϵ_t are i.i.d. with $E(\epsilon_t) = 0$ and $E(\epsilon_t^2) = 1$.

Robinson and Zaffaroni (1997) discuss LRD parameterizations for the ψ_j in (2). An important parametric version is

$$1 - \sum_{j=1}^{\infty} \psi_j z^j = (1 - z)^d \frac{b(z)}{a(z)}, \quad (5)$$

where $d \geq 0$ and $a(z)$ and $b(z)$ are defined as finite order polynomial whose roots are outside the unit circle in the complex plane. Setting $\psi(z) = 1 - \sum_{j=1}^{\infty} \psi_j z^j$ and $\phi = \tau^2 \psi(1)$, (2) can be rewritten as

$$\sigma_t^2 = \phi + (1 - \psi(L))x_t^2.$$

Note that in this case (5) implies $\phi = 0$.

Relaxing the condition $E(\epsilon_t^2) = 1$, Giraitis *et al.* (2000) show that when

$$(E(\epsilon_t^4))^{1/2} \sum_{i=1}^{\infty} \psi_i < 1, \quad (6)$$

there exists a strictly stationary solution to (2) with bounded fourth moment. (6) allows for hyperbolically decaying ψ_j , although parameterizations such as (5) are excluded when $E(\epsilon_t^2) = 1$. Moreover, they show that (6) implies summability of the ACF of the x_t^2 , ruling out LRD.

The ARCH(∞) (2)-(4) with the parameterization (5) was considered in Baillie *et al.* (1996) but letting ϕ be a free parameter. This implies that when both $d, \phi > 0$ (and $E(\epsilon_t^2) = 1$) the x_t^2 are not covariance stationary. They propose Gaussian pseudo-maximum likelihood estimation (PMLE) of the model. However, no formal asymptotic distribution theory yet exists for the PLME nor for exact MLE (making distributional assumptions on the rescaled innovations ϵ_t). This holds irrespective of whether ϕ is considered a free positive parameter or not and for any chosen finite parameterization of the $\psi_j = \psi_j(\theta)$, θ denoting a p -dimensional parameter, including (5).

Indeed, even when focusing on exponentially decaying ψ_j , the asymptotic properties of the Gaussian PMLE are known only for the ARCH(p) (see Weiss (1986)) and

the GARCH(1, 1) (see Lee and Hansen (1994) and Lumsdaine (1996)), obtained when $p = 2$ and $\psi_j = \alpha\beta^{j-1}$ ($j \geq 1$) for some nonnegative parameters α, β satisfying the strict stationarity condition, $E \log(\beta + \alpha\epsilon_t^2) < 0$ (see Nelson (1990)). In a linear long memory semiparametric framework, Robinson and Henry (1999) show that the asymptotic properties of the local Whittle estimator Künsch (1987) are unchanged when relaxing conditional homoskedasticity of the linear innovations to ARCH(∞) dynamic conditional heteroskedasticity.

The ARCH(∞), and more generally ARCH-type models, can be seen as nonlinear autoregressive models. One can also consider nonlinear moving average (MA) models. In order to take into account the leverage effect and to relax nonnegativity constraints on the conditional variance parameters, Nelson (1991) introduced the EGARCH, given by (4) and

$$\log \sigma_t^2 = \omega + \sum_{k=1}^{\infty} \beta_k g(\epsilon_{t-k}), \quad \sum_{k=1}^{\infty} \beta_k^2 < \infty. \quad (7)$$

The scalar function $g(\cdot)$ defines the so-called news impact curve

$$g(\epsilon_t) = \theta \epsilon_t + \gamma(|\epsilon_t| - E|\epsilon_t|),$$

for scalar parameters γ, θ . Dynamic asymmetry requires $\theta \neq 0$ and identifiability $\beta_1 = 1$. Nelson (1991) mentioned the possibility of long memory parameterizations for the β_k , further developed in Bollerslev and Mikkelsen (1996) who proposed estimation of the model by Gaussian PMLE. However the asymptotic properties of the Gaussian PMLE are unknown both for short and long memory parameterization of the EGARCH model.

An alternative nonlinear MA has been proposed by Robinson and Zaffaroni (1997):

$$x_t = \epsilon_t h_t, \quad (8)$$

$$h_t = \rho + \sum_{j=1}^{\infty} \alpha_j \epsilon_{t-j}, \quad \sum_{j=1}^{\infty} \alpha_j^2 < \infty, \quad (9)$$

generalizing the nonlinear MA(1) of Robinson (1977). This yields $\sigma_t^2 = \text{var}(\epsilon_t)h_t^2$. The choice of the α_i defines the memory properties of the x_t^2 , which are weaker or equal than the ones of the h_t depending on whether ρ is equal to zero or not. Focusing on a wide class of long memory parameterization, including the case where the α_j are the moving average coefficients of the ARFIMA, Zaffaroni (1999b) establishes $n^{1/2}$ -consistency (n denotes sample size) and asymptotic normality of the Whittle estimator of the model, when fitting the spectral density of the squares x_t^2 .

The nonlinear AR and MA models just described can be expressed as nonlinear transformations of present and lagged values of the ϵ_t . Therefore, these models could be defined as one-shock models. In principle, one can invert the models and express the unobservable innovation, the ϵ_t , as a nonlinear function of present and lagged values of the observable, the x_t , although formal proofs of invertibility may prove very difficult to establish.

Motivated by the fact that they seem to provide a closer approximation to continuous time pricing formulae of modern finance, Stochastic Volatility (SV) models, introduced by Taylor (1986), replace (4) with

$$x_t = \eta_t \sigma_t, \quad (10)$$

and setting

$$\log \sigma_t^2 = \mu_0 + \sum_{k=1}^{\infty} \mu_k \epsilon_{t-k}, \quad \sum_{k=0}^{\infty} \mu_k^2 < \infty,$$

where $\{\eta_t\}$ represents an i.i.d. sequence with $E(\eta_t) = 0$, $var(\eta_t) = 1$, different from the ϵ_t . Hence, SV-type models can be defined as two-shock models. In most cases it is assumed that η_t and ϵ_t are independent of one another, although this rules out the possibility to account for dynamic asymmetries. SV challenge ARCH-type models in modelling the empirical distribution of asset returns, see Ghysels *et al.* (1996) for a complete survey. Although the decoupling of the two shocks makes SV more attractive for moment formulae, the latent nature of the conditional variance complicates the evaluation of the conditional variance through filtering and, more importantly, of the likelihood function itself.

A long memory SV model was introduced by Harvey (1993), setting

$$\log \sigma_t^2 - \mu_0 = (1 - L)^{-d} \epsilon_{t-1}, \quad 0 < d < 1/2,$$

and proposed estimation by a discrete Whittle estimator, fitting the model spectral density of $\log x_t^2$. Breidt *et al.* (1998) show consistency of this estimator and Deo and Hurvich (1999) show asymptotic normality of the GPH estimator for this model. Comte and Renault (1998) discuss statistical properties and temporal aggregation issues relative to a continuous time long memory SV model.

A two-shock nonlinear MA was introduced in Robinson and Zaffaroni (1998) which allows time-varying first and second conditional moments. It is defined by

$$x_t = g_t + \eta_t h_t,$$

where the bivariate process g_t, h_t is independent of the η_t , defined in (10). The memory properties of the x_t and x_t^2 are characterized for a general specification of the g_t, h_t . The linear specification, with h_t given by (9) and $g_t = \mu + \sum_{i=0}^{\infty} \beta_i \epsilon_{t-i}$, $\sum_{i=0}^{\infty} \beta_i^2 < \infty$, seems the most appealing specification, prior to finite parameter modelling for the purpose of estimation.

Focusing on Gaussian ϵ_t and η_t , Robinson (1999) shows how all these SV-type models can be nested in a general framework, setting $x_t = f(\delta_t)$ where δ_t is a q -dimensional stationary Gaussian process and $f(\cdot)$ an arbitrary function $f : R^q \rightarrow R$ such that $E(f^2(\delta_t)) < \infty$. An asymptotic expansion for the ACF of the x_t so defined is provided, which allows to distinguish the impact of the nonlinear transformation $f(\cdot)$ and of the memory of the δ_t , respectively, on the memory of the x_t . In particular, assuming the δ_t exhibit LRD, primitive conditions on the function $f(\cdot)$ are established such that the x_t would exhibit the same or a weaker degree of LRD, respectively.

Despite the numerous LRD volatility models so far proposed, very little is known in terms of the possible sources of LRD in squared asset returns. Inspired by the corresponding linear time series analysis, the contemporaneous aggregation mechanism seems to provide an important vehicle. Ding and Granger (1996) and Andersen and Bollerslev (1997) employ the linear aggregation results to a GARCH and SV setting, respectively. In both cases, the aggregate is not truly defined as the cross-sectional arithmetic average. In a strong GARCH setting, Zaffaroni (1999a) establishes the probabilistic properties of the aggregate, defined as the cross-sectional arithmetic average of heterogeneous GARCH. In particular, it is shown that the aggregation mechanism never induces LRD in the conditional variance of the aggregate.

2.3 Pricing implications.

The principle of non-arbitrage is the corner-stone of all pricing models of modern finance,

since the seminal work of Harrison and Kreps (1979). Mandelbrot (1971) first analyzed the pricing implication of long memory. In a single asset framework, he assumed that the first difference of the fundamental follows a linear process, $P_0(t) - P_0(t-1) = \sum_{i=0}^{\infty} \alpha_i N(t-i)$, for some i.i.d. white noise sequence $\{N(t)\}$ and $\sum_{i=1}^{\infty} \alpha_i^2 < \infty$. Further, it was assumed that arbitrage would induce the speculative price of the asset, $P(t)$, to be a linear function of the fundamental and to display the martingale property, implying $P(t) - P(t-1) = N(t)(\sum_{i=0}^{\infty} \alpha_i)$ under risk neutrality and a zero risk-free interest rate. Hence, LRD of the differenced fundamental, i.e. non-summability of the α_i , directly implies the impossibility of implementing arbitrage.

Although with ambiguous results (cf. §2.1), the empirical research that focused on the possibility of LRD in the levels of asset returns prompted the relevance of the standard Brownian motion model in favour of fractional Brownian motion. However, Maheswaran and Sims (1993) first pointed out that fractional Brownian motion of type $B_H(t)$ ($0 < H < 1, 0 \leq t \leq 1$) is not a feasible model for asset returns. In fact, $B_H(t)$ is not a semimartingale unless $H = 1/2$ for which $B_{1/2}(t) = B(t)$. For the case $1/2 \leq H < 1$ see also Cutland *et al.* (1995) and more recently Shiryaev (1998). Rogers (1997) extended the result to the case $0 < H < 1/2$.

Without the semimartingale property, the standard approach for pricing contingent claims, based on the equivalent martingale measure, fails. This result stems from the fact that the order- q variation of $B_H(t)$ satisfies (cf. (Rogers, 1997, eq. (2.2))), as $n \rightarrow \infty$,

$$\sum_{j=1}^{2^n} |B_H(j2^{-n}) - B_H((j-1)2^{-n})|^q \rightarrow_p \begin{cases} 0, & qH > 1, \\ \infty, & qH < 1, \end{cases}$$

consistent with a semimartingale behaviour for $H = 1/2$ only. Rogers (1997) shows that the failure of the semimartingale property characterizes all Gaussian processes $\{X(t)\}$ of the form

$$X(t) = \int_{-\infty}^t \phi(t-s)dB(s) - \int_{-\infty}^0 \phi(-s)dB(s), \quad t \in R^+,$$

with square integrable kernel $\phi(\cdot)$ satisfying $\phi(t) \sim c t^{H-1/2}$ as $t \rightarrow 0^+$. $B_H(t)$ represents a particular case setting $\phi(t) = t^{H-1/2}$. As a consequence, it follows that choosing $\phi(t)$ such that $\phi(0) = 1$, $\phi'(0) = 0$ and $0 < \phi''(t)t^{5/2-H} = O(1)$, $t \rightarrow \infty$, one can construct Gaussian processes which exhibit the same LRD as $B_H(t)$ while retaining the semimartingale property (Rogers, 1997, section 5). Salopek (1998) addressed the issue of arbitrage opportunities in a more general framework, with respect to the class of processes with continuous sample paths and bounded q -variation ($1 \leq q < 2$). Cutland *et al.* (1995) and Shiryaev (1998) discuss fractional version of the Black-Scholes option pricing formula, based on $B_H(t)$. These developments, which use $B_H(t)$ for describing dynamics of asset prices, could still be useful, though, as the corner-stone assumption of zero transaction costs is likely to fail in practice.

When considering the possibility of LRD in squared returns with martingale difference levels (cf. §2.2), the mentioned implications in terms of the possibility of arbitrage do not apply. Therefore the standard risk-neutral valuation methods can be used. Comte and Renault (1998) discuss option pricing formulae based on a continuous time long memory SV model.

On the other hand, one must be aware of these implications when considering volatility models characterized by a time-varying conditional mean specified, since Engle *et al.* (1987), as a linear function of the square-rooted conditional variance when the latter displays long memory. Mandelbrot *et al.* (1997) and Heyde (1999) proposed continuous

time subordinated process for financial asset returns, apt to display heavy tails and LRD in absolute-valued and squared returns.

Practical use of long memory asset pricing formulae in empirical applications is not so widespread yet. Using daily data of exchange traded long-term equity anticipation securities on the *S&P 500* index, Bollerslev and Mikkelsen (1999) propose an empirical applications of pricing long term options, parameterizing the conditional variance with the so-called fractionally integrated GARCH ((7) coupled with $\sum_{k=0}^{\infty} \beta_k L^k = \psi^{-1}(L)$ as from (5)).

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