

About seasonal adjustment: a univariate and multivariate framework

Sull'aggiustamento stagionale: un approccio univariato e multivariato

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Riassunto: In assenza di consenso circa i modelli strutturali più appropriati a descrivere la stagionalità dei dati economici, si propone una metodologia di aggiustamento stagionale basata su un semplice approccio lineare e descrittivo ai processi integrati. L'unica assunzione richiesta è che il polinomio associato alla rappresentazione di Wold del processo propriamente differenziato soddisfi una certa condizione di sommabilità. L'idea di base consiste nel rimuovere dai dati grezzi le passeggiate aleatorie stagionali associate ad ogni frequenza stagionale. Questo approccio consente tra l'altro l'aggiustamento stagionale simultaneo di un insieme di variabili e l'uso di procedure di stime robuste alla presenza di valori anomali.

Keywords: Seasonal Adjustment, Seasonally Integrated Processes

1 Introduction

The demand for seasonally adjusted time series comes from the fact that raw data are difficult to read : seasonal adjustment procedures used by statistical agencies are supposed to remove intra annual regularities, to leave the users with a clearer view of the underlying movements of the series.

This simple statement is in contrast with the fact that statisticians do not agree on a definition of the seasonal component of a time series since any rigorous statistical definition can only be based on a prior model of the series, and there is no agreement on a common model between the different users or the statisticians. There are very scarce studies in which the seasonal factors are modeled per se, and no agreement in the profession as to the best treatment(s) of the phenomenon.

In this context, statisticians have favored a descriptive approach making the pragmatic observation that the reading of raw data is made difficult by the presence of persistent components at seasonal frequencies. These components by the fact that they repeat similar patterns each year hide the long and medium term changes and the statistical treatment aims at removing them with as small as possible distortions at the other frequencies. The simplest case of persistent seasonal components at frequency $\omega \in]0, \pi[$ is the linear combination of the deterministic functions $\sin \omega t$ and $\cos \omega t$. More flexible processes that allow for evolving oscillating patterns can be built with complex random walks at the same frequencies. The definition of the seasonal persistent components is then a key ingredient of the models used to describe the series. The usual model assumes that the series under study can be decomposed into the sum of uncorrelated components, one of them generating the seasonal pattern. This supposes that the sources of seasonality do not interact with the sources of medium and long term changes. We follow here a different route. We keep to a descriptive and additive decomposition but in which the innovations

of each persistent component are perfectly correlated, meaning changes in the seasonal pattern and in medium and long term have common sources. This decomposition is natural and always exists for any integrated process of various orders at various frequencies. Different frameworks can be chosen to estimate these components. We present in a weak linear framework a method to estimate them for any process satisfying a SARIMA representation.

The demand for seasonally adjusted series takes various forms, depending in particular on the time series background of the user: from the general public who may not be aware of the existence of raw data to sophisticated econometricians who specify the model of seasonals in their studies, there is a continuum of situations. When the demand comes from practitioners, i.e. short-term analysts in banks and government agencies, journalists or policy makers, the seasonally adjusted data are used to compare the state of the economy at different points in time or to compare the situation of different economies, by looking at a comprehensive set of statistics, including production, household purchases, employment, credit, wages and prices... Once corrected of regular seasonal movements, the data allow them to propose an interpretation of the mechanisms at work that can explain the observed co-movements. These interpretations may govern investment decisions or economic policy changes. In respect with this objective, it seems that current practices fail to meet the needs of most users : the usual seasonal adjustment statistical procedures are univariate. In practice for dissemination purpose, it is not possible to provide the various users with different sets of simultaneously seasonally adjusted time series according to peculiar sets of information. It might nevertheless be of interest to provide them with raw data to allow them to proceed to this treatment by themselves or with some standard sets of simultaneously seasonally adjusted data. The framework we introduce here allows us to compute such a seasonal adjustment for multivariate time series.

The trend towards flexibility we observe in the seasonal adjustment packages corresponds to a need to adapt descriptive models to an ever changing economic environment. It shows through the increasing number of options given to the users. It sometimes may introduce some instability in the estimates of the seasonally adjusted time series. Facing the same difficulties, the linear framework introduced here can naturally be adapted to robust estimation to control the impact of outliers. It also allows the statistician to provide the users with confidence interval on the estimate of the seasonally adjusted time series. One key question in this approach is nevertheless related to the possibility to select the appropriate model to describe the persistent components at seasonal frequencies. From a statistical point of view, this corresponds to the possibility to implement efficient tests for the presence of seasonal unit roots to capture evolving seasonal patterns in possible presence of deterministic seasonal functions. Recent developments provide us with such tests. One limit of the framework is related to the fact that the correction it induces amounts to the use of an asymmetric filter that may induce some phase shift.

The following section deals with the sources of seasonal patterns in economic data and with a brief presentation of the seasonal adjustment procedure principles. Some notations and useful algebra results are given in the third section, they allow us to characterize and build up a persistent process at a seasonal frequency. The fourth section is dedicated to the decomposition of any univariate or multivariate integrated process into a sum of persistent processes at all the frequencies associated to their unit roots and a transitory component. The fifth section deals with the practical implementation of the method for SARIMA univariate processes and multivariate seasonally cointegrated ones. A short conclusion closes the paper.

2 Seasonality

2.1 Various sources¹

2.1.1 Natural and institutional causes

Seasonal patterns in economic time series result from exogenous sources and their interactions with economic agent rational choices.

The exogenous sources of intra-annual regular changes in economic time series can be classified into three main categories. The main source of seasonal pattern is climatology and biology related. Household consumption in energy products, fresh fruits or fishes, or ski resort services among others are clearly motioned by weather conditions and biological cycles. Agricultural production, as well as construction, are clearly affected by the season. The participants in the economy adapt to these exogenous shocks. Consumers buy expensive imported products from the other hemisphere. There are insurance provisions for workers in the building industry to deal with the cold weather.

The second source of seasonal pattern in economic time series comes from society. Religion or custom motivate shifts in demand during the year. For instance, the gift-giving period at the end of the calendar year in the main developed countries leads to a large increase in consumption in December and pushes upward production and storage during the fourth quarter (Beaulieu and Miron (1992)). By law, the minimum wage and the minimum pension are revised at some specific predetermined dates in the year. Likewise, Eastertide is associated with a decrease in industry and banking activity and a marked increase in household consumption of particular products such as chocolate.

A third source is linked to the division of time in years, months, weeks and its interaction with the two preceding sources. Labor market seasonal patterns are related to institutional beginning and end of school period. Bonus or “13th month” payments are usually paid at the end of the year and induce a large increase in aggregate wages. More generally, the average number of weekdays and week-ends propitious to production or shopping varies with the calendar month. In developed economies, a large part of household purchase takes place on Saturday, which partly contributes to the variability of the monthly growth rates of household consumption.

2.1.2 Compilation by the statistical agencies

Seasonal patterns may also be related to the way statistical agencies compile their statistics. The samples of the infra annual surveys typically are not designed to trace accurately the seasonal patterns. Seasonal changes in the compiled figures reflect a seasonal change of the activity of the surveyed population which does not necessarily mimic the variations of the national aggregate. For instance, consider the case in which household expenditure is measured by the sales at a fixed sample of stores. The store population is stratified according to some structural criteria (last year sales, number of employees,...), but is typically not based on seasonal patterns. Aggregate figures are computed using the weights of the sample, for instance, by multiplying the monthly growth rate of the sales of the strata by the ratio of the previous year sales of the strata to the total. During their holidays, households may leave their usual surroundings and go to the seaside or winter sport resorts. Since the weights are usually computed from *yearly* turnover, the stores

¹This section draws on Gregoir and Laroque (1998).

in the sample which belong to the holiday resorts are overweight during the non-holiday seasons and underweight in vacation time. Since the sample is not explicitly designed to reproduce the seasonal changes, there is no guarantee that the seasonal pattern in the aggregate time series be anything but artificial.

It is worth emphasizing that the seasonal component of an economic time series typically is not in a simple relationship with the exogenous seasonal shocks. These shocks are transformed, smoothed, mixed with other shocks, by the economic agents whose decisions presumably maximize an intertemporal objective. Removing the seasonal component therefore takes off some economic information of interest. As stated by Bell and Hillmer (1984), a seasonal adjustment procedure must aim at simplifying the data while limiting the loss of information.

2.2 Various treatments

In the absence of a consensus on the appropriate structural models that would allow the statisticians to remove from the data the persistent seasonal components (see Ghysels (1988) for an illustration of the inefficiency of standard procedures in a completely specified framework), today seasonal adjustment procedures are mainly derived from a descriptive framework. In all these procedures, the maintained assumption is the unobserved component model : the time series under study is well described as the sum of several unobserved and uncorrelated components, one of which being seasonal. Most of them use parametric modeling to allow for classical inference and possible use of optimal statistical signal extraction, but they mainly differ in their choices and use of these models.

Two main classes of models are considered. On the one hand, the structural time series models postulate some a priori parameterized components (see *inter alios* Engle (1978), Harvey and Todd (1983), Harvey (1989) and in a bayesian framework, Akaike (1980), Kitagawa and Gersch (1984)). On the other hand, the ARIMA model family is used for backcasting and forecasting at the beginning and end of the sample when one wants to use symmetric filters (Dagum (1975, 1978)) and for getting a decomposition of the observed time series into its components. The seasonal component is then essentially identified by the assumption of no correlation at all leads and lags with the remaining components. SARIMA models are posited for the overall series with constraints on the location of the roots associated to their polynomials and signal extraction based on Wiener-Kolmogorov filters yields the desired components under some identifying restrictions (see *inter alios* Box, Hillmer and Tiao (1978), Burman (1980), Hillmer and Tiao (1982), Gómez and Maravall (1996)).

In practice, the estimation of a SARIMA model has several steps. At the beginning, one has to determine the order of integration at the various frequencies. On the one hand, interactions between the deterministic part of the process and the presence of (seasonal) unit roots and on the other hand preliminary statistical treatments aiming at removing exogenous (i.e. working days) factors may make the analysis particularly delicate. Usually, first-difference and seasonal first-difference operators are applied *a priori*, which may lead to over-differentiation². In a second step, the statistician applies a variant of the Box and Jenkins (1970) approach to fit a SARMA model. To prevent the procedures to become overly time consuming, a number of default options and rules of thumb are

²This choice is also related to the a priori parameterization of the components in the ARIMA model based procedure.

introduced which are particularly useful when a large number of times series must be treated.

This calls for comments. On the one hand, it can be stressed that the above decomposition is not always possible as shown and illustrated by Hillmer and Tiao (1984). On the other hand, the changing economic environment makes it unlikely that a given model fits a time series for a long span, even though some kind of parameter instability is allowed for through the treatment of outliers. From a practical perspective, exploratory descriptive measures are often provided to assess the reliability (stability and robustness) of the results and help selecting the “appropriate” filters. For instance the X-12-ARIMA package gives sliding-span (Findley, Mosell, Shulman and Pugh (1990)), and revisions history (Findley, Monsell, Bell, Otto and Chen (1998)) diagnosis. These descriptive measures can be very informative, but their use is necessarily subjective.

The univariate available seasonal adjustment procedures which are multistep have shown a trend towards more flexibility multiplying the options. It lies in the choice of multiplicative vs. additive adjustment, in the detection of outliers or the specification of structural breaks, in the forecasting and backcasting model of the raw series for the estimation of the current points, or more fundamentally in the selection and computation of the seasonal filter. This may induce possible instability and therefore a difficult reading of the end of sample data or of the comovements between various variables. Indeed, the current data point is looked at with most attention, but it is unfortunately the less reliable for at least two reasons : it is subject to preliminary-data error and it is obtained by applying either a one sided filter, or a two sided filter to a forecast of the raw data. Therefore it necessarily is subject to revisions as time goes by. The revisions are particularly important when an outlier observation is present towards the end of the series. For outliers with an assignable cause, an intervention analysis in the spirit of Box and Tiao (1975) seems to be relevant but may be costly and *de facto* rarely used. In practice, outliers are identified with respect to the selected model. This identification is contingent upon the sample to hand and not necessarily time consistent. An observation detected as an outlier for a given sample size may appear as a regular observation in a larger sample. Similarly, concurrent adjustment³ may also induce noisy revisions if the selection of the forecasting and backcasting model is erratic and not tightly controlled. At last, since the interest is in multiple series and the practice is to deal with each series separately, flexibility creates the risk of introducing differences between series due to different choices of seasonal adjustment procedures and, when these series must be added into an aggregate, may affect the behavior of the seasonally adjusted aggregate. Sims (1974) and Wallis (1974) pleaded for the use of the same linear filter to all series appearing in a multiple regression.

An ideal framework would provide the statistician with among other things a always possible decomposition into seasonal and non-seasonal components, seasonally adjusted time series as close as possible to the raw one at all the frequencies but the seasonal ones, with the smallest shift in phase, in a set-up that allows for robust estimation in order to be less sensitive to outlier definition or characterization.

The linear framework we introduce below meets some of these requirements. Moreover it allows for the computation of confidence intervals for the seasonally adjusted processes and for the treatment of multivariate time series. Its main drawback lies in the fact that the filter we obtain is model dependent and may induce a phase shift.

³A couple of studies in the literature advocate ‘concurrent’ adjustment, i.e. calculation of seasonal factors each time new data become available even for X11 procedure (Kenny and Durbin [1982], Wallis [1982], [1983], McKenzie [1984], Pierce and McKenzie [1987]).

3 Persistence at seasonal frequency

3.1 Integral operator algebra

We consider the integrated processes at various frequencies. They are such that simple linear operators make them covariance stationary processes, analysis of which can be carried out with the help of their autocorrelation structure or their spectrum. We start by introducing some notation and restating a well-known definition. In the sequel, the real and complex first-difference operators at various frequencies are denoted as follows:

$$\Delta_\omega(B) = \begin{cases} \delta_\omega(B) = 1 - e^{-i\omega}B, & \omega \in \{0, \pi\} \\ 1 - 2\cos\omega B + B^2 = (1 - e^{-i\omega}B)(1 - e^{i\omega}B) = \delta_\omega(B)\delta_{-\omega}(B) & \omega \in]0, \pi[\end{cases}$$

where $\omega \in]0, \pi[$ and B is the backshift operator. When there is no ambiguity, we sometimes replace $\Delta_\omega(B)$ or $\delta_\omega(B)$ by Δ_ω or δ_ω .

Definition 1 A purely non deterministic real random process $\{x_t\}$ is said to be integrated of order h , h integer, at frequency ω , $\omega \in [0, \pi]$, or $I_\omega(h)$, if $\Delta_\omega(B)^h x_t$ is a covariance stationary process such that the series $c(u)$ associated with its Wold representation satisfies $c(e^{i\omega}) \neq 0$ (consequently $c(e^{-i\omega}) \neq 0$ when $\omega \in]0, \pi[$).

Remark 2 The non zero condition on the value at frequency ω of the polynomial associated to the Wold representation can be rephrased as a condition of a non zero spectral density at frequency ω .

Remark 3 A complex valued process is integrated of order h at frequency $\omega \in [0, \pi]$ if the application of the operator δ_ω^h gives a covariance stationary process with a non-zero value of the polynomial associated to its Wold representation at frequency ω .

In practice, processes can be integrated simultaneously at various frequencies. In this case, the product of the related first-difference operators raised at the appropriate power have to be applied to obtain a covariance stationary process. Following Gregoir(1999a), we now introduce a set of integration operators that play the role of the inverses of the first difference operators at each frequency. They are particularly well adapted to introduce the unobserved component decomposition we propose to use here.

Definition 4 For any sequence $y_t = (y_t, t = \dots, -1, 0, 1, \dots)$ of $\mathbb{R}^Z$ the integral operator S_ω associates a sequence $S_\omega y_t$ defined by:

$$S_\omega y_t = \begin{cases} \sum_{\tau=1}^t y_\tau e^{-i\omega(t-\tau)} & t \geq 1 \\ 0 & t = 0 \\ -\sum_{\tau=t+1}^0 y_\tau e^{-i\omega(t+1-\tau)} & t < 0. \end{cases}$$

To understand the role played by these integral operators, apply one of them to a real white noise ϵ_τ . This can be decomposed into three steps. The first one corresponds to the definition of a complex white noise $e^{i\tau\omega}\epsilon_\tau$, the second one to the construction of a $I_0(1)$ complex valued process: $\sum_{\tau=1}^t e^{i\omega\tau}\epsilon_\tau$ for $t \geq 1$ (and the appropriate operation for $t \leq 0$). Lastly, a demodulation operation as described by Granger and Hatanaka (1967) (the multiplication by $e^{-i\omega t}$) shifts the peak of spectrum of the $I_0(1)$ process to the frequency ω . This therefore gives a $I_\omega(1)$ complex valued process. Applied to the same

white noise, different integral operators have an empirical covariance that asymptotically converges to zero. If we want to avoid complex numbers. The right way to build an $I_\omega(1)$ time series when $\omega \in]0, \pi[$ is to apply to an $I(0)$ series the operator $S_\omega S_{-\omega}$. Simple algebraic computations give :

$$S_\omega S_{-\omega} y_t = \begin{cases} \sum_{\tau=1}^t y_\tau \frac{\sin \omega(t+1-\tau)}{\sin \omega} & t \geq 1 \\ 0 & t = 0 \\ -\sum_{\tau=0}^{t+1} y_\tau \frac{\sin \omega(t+2-\tau)}{\sin \omega} & t < 0. \end{cases}$$

These operators satisfy a set of algebraic properties that are related to the fact that we consider cumulative operations at various frequencies. We list some of them which are used in the sequel (cf Gregoir (1999a)).

$$\begin{aligned} \forall \omega \in [-\pi, \pi], \\ \delta_\omega S_\omega = I \end{aligned} \tag{1}$$

$$\begin{aligned} \forall y_t \in \mathbb{R} \\ \omega \in [0, \pi], S_\omega \delta_\omega y_t = y_t - y_0 e^{-i\omega t} \end{aligned} \tag{2}$$

$$\omega \in]0, \pi[, S_\omega S_{-\omega} \Delta_\omega y_t = y_t - y_0 \frac{\sin \omega(t+1)}{\sin \omega} + y_{-1} \frac{\sin \omega t}{\sin \omega}$$

$$\begin{aligned} \text{(Commutativity)} \quad \forall (\omega_1, \omega_2) \in [0, \pi]^2, \\ S_{\omega_1} S_{\omega_2} = S_{\omega_2} S_{\omega_1} \end{aligned} \tag{3}$$

$$\begin{aligned} \text{(Associativity)} \quad \forall (\omega_1, \omega_2, \omega_3) \in [0, \pi]^3, \\ (S_{\omega_1} S_{\omega_2}) S_{\omega_3} = S_{\omega_1} (S_{\omega_2} S_{\omega_3}) \end{aligned} \tag{4}$$

$$\begin{aligned} \forall (\omega_1, \omega_2) \in [0, \pi]^2, \\ S_{\omega_1} S_{\omega_2} = \frac{e^{i\omega_1}}{e^{i\omega_1} - e^{i\omega_2}} S_{\omega_2} + \frac{e^{i\omega_2}}{e^{i\omega_2} - e^{i\omega_1}} S_{\omega_1} \end{aligned} \tag{5}$$

From equations (1) and (2), we observe that the two types of operators do not commute. The deterministic functions we introduce in (2) result from the definition of the integral operators. The integrated processes we construct with them are all equal to 0 at date $t = 0$. This a convention. A different convention would lead to an other deterministic function in (2). The last item of this list allows us to replace any product of integral operators at different frequencies in a sum of polynomials in operators at each frequency. Here are some illustrations

Example 5 $\forall \omega \in]0, \pi[$,

$$\begin{aligned} S_\omega S_{-\omega} &= \frac{e^{-i\omega}}{e^{-i\omega} - e^{i\omega}} S_{-\omega} + \frac{e^{i\omega}}{e^{i\omega} - e^{-i\omega}} S_\omega \\ S_0 S_\omega S_{-\omega} &= \frac{1}{2(1 - \cos \omega)} S_0 + \frac{1 - 2\cos \omega + B}{2(1 - \cos \omega)} S_\omega S_{-\omega} \end{aligned}$$

$$\forall (\omega_1, \omega_2) \in]0, \pi]^2, \omega_1 \neq \omega_2,$$

$$S_{\omega_1} S_{-\omega_1} S_{\omega_2} S_{-\omega_2} = \frac{2\cos \omega_1 - B}{2(\cos \omega_1 - \cos \omega_2)} S_{\omega_1} S_{-\omega_1} + \frac{2\cos \omega_2 - B}{2(\cos \omega_2 - \cos \omega_1)} S_{\omega_2} S_{-\omega_2}$$

3.2 Persistent components

The possibility to describe parsimoniously a large set of macroeconomic time series as integrated ones has allowed for the introduction of new relevant descriptive statistics. If the use of local time descriptive statistics (Phillips (1998)) has not yet attracted a great deal of attention, the measure of persistence has been the subject of various studies (Campbell and Mankiw (1987), Cochrane (1988)). Persistence of shocks at frequency 0 is related to the fact that contrary to covariance stationary time series, integrated processes are such that the effect of their innovations does not vanish asymptotically but persists. Empirical macroeconomists propose various measures of the magnitude of this persistence in considering decompositions into the sum of a permanent and a transitory components. Nevertheless Quah (1994) shows that the measure of persistence by itself is dependent on the choice of these two component models and may not be so informative in a structural point of view. In particular, the underlying permanent component can be chosen to be arbitrarily smooth so that the transitory component may dominate the fluctuations of the series at all finite horizons. From a descriptive point of view, various decompositions in unobserved components can be considered. Watson (1986) introduces three basic situations : when the persistent and transitory unobserved components are two orthogonal processes, when their respective innovations are perfectly correlated and a third situation in between. He notes that if the first decomposition imposes constraint on the shape of the spectral density of the covariance stationary process obtained by first differencing, the third one necessitates a basis for the choice of the two component models that may refer to some structural considerations. The second one is always possible but may sometimes give a persistent component estimate very close to the original series which may be a limit to its usefulness.

We can transpose this analysis to persistent seasonal processes corresponding to processes integrated at frequency $\omega \in]0, \pi]$. As said above, the difficult reading of seasonal time series comes from the persistent seasonal components present in the time series. A natural strategy for seasonal adjustment consists in decomposing the seasonal time series into a sum of persistent components at each frequency and in removing the appropriate ones. In this section, we extend a characterization of persistent time series at a given frequency different from 0. In the next section, we introduce the decomposition. As indicated above, the definition of the persistent component cannot be unique. Considerations on simplicity and flexibility will lead to our selection of a always valid definition of persistent seasonal components. Alternative models might be considered and would produce different seasonally adjusted time series.

We extend a characterization of persistence to frequency different from zero by considering a demodulation operation at frequency ω . Doing this, we shift the spectrum along the frequency axis by ω and consider $I_0(1)$ complex process that can be decomposed into a persistent and a transitory component as usual. This decomposition is available for a process that is integrated at only one given frequency. If the process is integrated at various frequencies a decomposition into different persistent components at each of these frequencies and a transitory component must be produced.

Remark 6 *Let $\{x_t\}$ be a purely nondeterministic (complex) time series integrated at only one frequency $\omega \in [0, \pi]$ then $\lim_{h \rightarrow +\infty} E_t e^{i\omega h} x_{t+h}$ where E_t stands for the linear expectation operator conditional on the available information up to date t is a non-zero constant term depending on this information.*

Example 7 The simplest case of persistent process at frequency $\omega \in [0, \pi]$ is the (complex) random walk :

$$x_t = e^{i\omega} x_{t-1} + \varepsilon_t$$

where $\{\varepsilon_t\}$ is a (complex) white noise. In this case, $\lim_{h \rightarrow +\infty} E_t e^{i\omega h} x_{t+h} = x_t$.

Example 8 Let $\{x_t\}$ be a covariance stationary process satisfying a AR(1) representation ($x_t = \rho x_{t-1} + \varepsilon_t$, $|\rho| < 1$), then $S_\omega x_t$ is a persistent process at frequency ω . By definition for $t > 0$

$$S_\omega x_{t+h} = \sum_{\tau=1}^{t+h} x_\tau e^{-i\omega(t+h-\tau)}$$

and therefore

$$e^{i\omega(t+h)} S_\omega x_{t+h} = \sum_{\tau=1}^{t+h} x_\tau e^{i\omega\tau}$$

so that

$$\begin{aligned} E_t e^{i\omega(t+h)} S_\omega x_{t+h} &= \sum_{\tau=1}^t x_\tau e^{i\omega\tau} + \sum_{\tau=t+1}^{t+h} E_t x_\tau e^{i\omega\tau} \\ &= \sum_{\tau=1}^t x_\tau e^{i\omega\tau} + \left(\sum_{\tau=t+1}^{t+h} \rho^{\tau-t} e^{i\omega\tau} \right) x_t \\ &= \sum_{\tau=1}^t x_\tau e^{i\omega\tau} + \frac{\rho e^{i\omega t} (1 - \rho^{h-1} e^{i\omega(h-1)})}{1 - \rho e^{i\omega}} x_t \end{aligned}$$

and

$$\lim_{h \rightarrow +\infty} E_t e^{i\omega h} S_\omega x_{t+h} = S_\omega x_t + \frac{\rho}{1 - \rho e^{i\omega}} x_t$$

4 Seasonal adjustment by subtraction of the seasonally persistent components

We now introduce the procedure we follow to decompose any integrated time series into a sum of a persistent seasonal component and a non-seasonal one, the persistent seasonal component being equal to the sum of deterministic seasonal functions and pure seasonal random walks. We consider a real time series, data generating process of which can be described as follows :

$$y_t = d_t + x_t$$

where d_t is a deterministic component which is a linear combination of deterministic functions $\{e^{iv_j t}\}_{v_j \in \Omega_d}$ with $\Omega_d = \{v_0, \dots, v_l\}$ and $\{x_t\}$ is an integrated of order one process at various frequencies in $\Omega_x = \{\omega_0, \dots, \omega_k\}$. Ω_d and Ω_x are subsets of the set $\{\tilde{\omega}_j = \frac{2j\pi}{s}, j \in \{0, 1, \dots, s\}\}$ of seasonal frequencies associated to the seasonal length s ,

the number of seasons in a year⁴. Let denote $\Delta_x(B) = \prod_{j=0}^k \delta_{\omega_j}(B)$ such that $\Delta_x(B) x_t$ is a covariance stationary process. By convention, we assume that $v_0 = 0$ and $\omega_0 = 0$, but the analysis can be extended without difficulties when the frequency $0 \notin \Omega_x$ or $0 \notin \Omega_d$ and similarly when the process $\{x_t\}$ is integrated of an order larger than 1. In our set-up, a seasonally adjusted time series is a time series without any persistent component at the seasonal frequencies, it is then obtained in subtracting persistent seasonal components from the raw data. We now introduce the framework first with integrated univariate time series then with multivariate ones and list some of the advantages and drawbacks of the approach.

4.1 Univariate processes

In short, the DGP of the process we consider is given by

$$\begin{cases} y_t = d_t + x_t \\ d_t = \sum_{j=0}^l \mu_{v_j} e^{i v_j t} \\ \Delta_x(B) x_t = c(B) \varepsilon_t \end{cases} \quad (6)$$

where the polynomial associated to its Wold representation is denoted $c(B)$ and the innovation process $\{\varepsilon_t\}$. We need an additional assumption to ensure that the decomposition we are going to introduce in the sequel involves covariance stationary transitory components. This restricts the polynomial associated to the Wold representation and rules out some case of fractional unit-root processes.

Condition 9 $c(B) = \sum_{p=0}^{+\infty} c_p B^p$ is such that $\sum_{p=0}^{+\infty} p |c_p| < +\infty$

Under Condition 9, there exists at each frequency $\omega_j \in \Omega_x$ a Beveridge-Nelson type decomposition of the polynomial $c(B)$ as follows

$$c(B) = c(e^{i\omega_j}) + \tilde{c}_{\omega_j}(B) (1 - e^{-i\omega_j} B)$$

where $\tilde{c}_{\omega_j}(B) = \sum_{p=0}^{+\infty} \tilde{c}_{\omega_j,p} B^p$ and $\tilde{c}_{\omega_j,p} = -\sum_{q=p+1}^{+\infty} c_q e^{i\omega_j(p-q)}$ such that $\{\tilde{c}_{\omega_j}(B) \varepsilon_t\}$ is a (complex) covariance stationary process.

From (1), we know that

$$\Delta_x(B) \prod_{j=0}^k S_{\omega_j} = 1 \quad (7)$$

introducing this equation in the DGP of $\{x_t\}$ gives

$$\Delta_x(B) \left(x_t - c(B) \left(\prod_{j=0}^k S_{\omega_j} \right) \varepsilon_t \right) = 0$$

On the one hand, there exists a deterministic function $\mu_{x,t} = \sum_{j=0}^k \mu_{x,\omega_j} e^{-i\omega_j t}$ such that $\Delta_x(B) \mu_{x,t} = 0$ and then

$$x_t = \mu_{x,t} + c(B) \left(\prod_{j=0}^k S_{\omega_j} \right) \varepsilon_t \quad (8)$$

⁴Since we limit our attention to real time series if $\omega_j \in]0, \pi[$ is an element of Ω_x , necessarily $2\pi - \omega_j$ is also in this set. Similarly, if $v_j \in]0, \pi[$ is an element of Ω_d , necessarily $2\pi - v_j$ is also in this set and the associated coefficients of the linear combination are conjugate.

On the other hand, from (5) there exists a set of $(k + 1)$ complex numbers⁵ $\{\phi_j\}_{j=0,\dots,k}$ such that

$$\prod_{j=0}^k S_{\omega_j} = \sum_{j=0}^k \phi_j S_{\omega_j} \quad (9)$$

The values of these $(k + 1)$ complex numbers can be derived from Bezout's Lemma. If we denote $\Delta_{x,-j}(B) = \prod_{m=0, m \neq j}^k \delta_{\omega_m}(B)$, from (7) and (9) we get

$$1 = \sum_{j=0}^k \phi_j \Delta_{x,-j}(B) \quad (10)$$

and therefore

$$\phi_j = \frac{1}{\Delta_{x,-j}(e^{i\omega_j})}$$

Equation (10) corresponds to the application of Bezout's Lemma to the family of coprime complex polynomials $\{\Delta_{x,-j}(B)\}_{j=0,\dots,k}$. Using the family of Beveridge-Nelson type decompositions, we get for each frequency

$$c(B) S_{\omega_j} \varepsilon_t = c(e^{i\omega_j}) S_{\omega_j} \varepsilon_t + \tilde{c}_{\omega_j}(B) \varepsilon_t$$

so that we can always propose a decomposition of $\{y_t, t \geq 0\}$ into the sum of a component without persistent seasonal components $\{y_t^*, t \geq 0\}$ and a component purely persistent at the seasonal frequencies $\{y_t^s, t \geq 0\}$ with the following definition

$$y_t = y_t^* + y_t^s$$

where

$$y_t^* = \mu_{v_0} + \mu_{x,\omega_0} + \phi_0 c(B) S_0 \varepsilon_t + \sum_{j=1}^k \phi_j \tilde{c}_{\omega_j}(B) \varepsilon_t$$

and

$$y_t^s = \sum_{j=1}^l \mu_{v_j} e^{iv_j t} + \sum_{j=1}^k \mu_{x,\omega_j} e^{-i\omega_j t} + \sum_{j=1}^k \phi_j c(e^{i\omega_j}) S_{\omega_j} \varepsilon_t$$

We notice from the definition of both components there exist several identification issues usual in the unobserved component framework we consider. They appear clearly at frequency 0 but can also be present if some $v_j \in \Omega_x$. They can be solved by introducing some *a priori* constraints on the properties of $\{x_t\}$. For instance when $\Omega_x = \Omega_d = \{0\}$, the constant term in the definition of y_t^* is the sum of two constants, one is related to the unobserved value of the process $\{x_t\}$ at date $t = 0$ and the other one is the difference between this value and the value y_0 . Any couple of two constants, sum of which is equal to y_0 is an admissible candidate. An additional assumption on x_0 must be considered. In the next section dealing with the implementation of this approach, this issue will be solved by the choice of the estimates we propose to consider.

⁵These complex numbers are conjugate for the couple of frequencies ω_j and $2\pi - \omega_j$.

The filter Ψ that is implicitly applied on the process $\{y_t - d_t\}$ to get the “seasonally adjusted” process $\{y_t^* - \mu_{v_0}\}$ is equal to

$$\Psi(B) = 1 - \sum_{j=1}^k \frac{\Delta_{x,-j}(B)}{\Delta_{x,-j}(e^{i\omega_j})} \frac{c(e^{i\omega_j})}{c(B)}$$

It is therefore equal to 1 in 0 and close to 1 in its neighborhood and equal to 0 at each frequency ω_j , $j = 1, \dots, k$. Nevertheless, the pseudo-spectrum of the “seasonally adjusted” process $\{y_t^* - \mu_{v_0}\}$ is equal to

$$f_{y^*}(\omega) = \left| \frac{1}{\Delta_{x,-0}(1)} \frac{c(e^{-i\omega})}{1 - e^{-i\omega}} + \sum_{j=1}^k \frac{\tilde{c}_{\omega_j}(e^{-i\omega})}{\Delta_{x,-j}(e^{ij\omega})} \right|^2 \frac{\sigma_\varepsilon^2}{2\pi}$$

The idea of decomposing a time series into unobserved components is usual in seasonal analysis. The more frequently used decomposition breaks a covariance time series into the sum of a trend, a seasonal and a noise components that are not correlated. In contrast with this approach, we determine here explicitly a component for each frequency. Such a decomposition was already considered by Hannan, Terrell and Tuckwell (1970) but under the assumption that the components are non correlated. In our framework, their innovations are equal. In their approach, this decomposition was possible for a subset of processes, here it is always available.

4.2 Multivariate processes

The framework we consider in the preceding section can be extended to multivariate processes without difficulties but at some notational cost. The same algebra of integral operators can be used to construct multivariate integrated processes at various frequencies. The main difference comes from the fact that we consider matrix polynomials rather than polynomials and rank considerations arise.

We consider a real n -dimensional time series, data generating process of which can be described as follows :

$$y_t = d_t + x_t$$

where d_t is a n -dimensional deterministic component which is a linear combination of deterministic functions $\{e^{iv_j t}\}_{v_j \in \Omega_d}$ with $\Omega_d = \{v_0, \dots, v_l\}$ and $\{x_t\}$ is a n -dimensional process such that its r^{th} component $\{x_{r,t}\}$ is a order one integrated process at various frequencies in $\Omega_{x,r} = \{\omega_0, \dots, \omega_{k_r}\}$. For sake of notational simplicity, we assume that each component is integrated at the same frequencies ($\Omega_{x,r} = \Omega_x$). This assumption can be removed without technical difficulties but at the cost of additional indices for each frequency, set of frequencies,.... Ω_d and Ω_x have the same properties as those in the preceding section. Let denote $\Delta_x(B) = \prod_{j=0}^k \delta_{\omega_j}(B)$ such that $\Delta_x(B) x_{r,t}$ is a covariance stationary process. $\Delta_x(B) x_t$ is a n -dimensional covariance stationary process whose polynomial matrix associated to its Wold representation is denoted $C(B)$ and its n -dimensional innovation process $\{\varepsilon_t\}$. The DGP we consider is therefore characterized by

$$\begin{cases} y_t = d_t + x_t \\ d_t = \sum_{j=0}^l \mu_{v_j} e^{iv_j t} \\ \Delta_x(B) x_t = C(B) \varepsilon_t \end{cases} \quad (11)$$

We again need an additional assumption to ensure that the decomposition we are going to introduce in the sequel involves only covariance stationary transitory components.

Condition 10 $C(B) = \sum_{p=0}^{+\infty} C_p B^p$ is such that $\sum_{p=0}^{+\infty} p (Tr C_p C_p')^{1/2} < +\infty$

We can follow the same lines of analysis as those in the preceding section. Under Condition (10), there exists at each frequency $\omega_j \in \Omega_x$ a Beveridge-Nelson type decomposition of the matrix polynomial $C(B)$ as follows

$$C(B) = C(e^{i\omega_j}) + \tilde{C}_{\omega_j}(B) (1 - e^{-i\omega_j} B)$$

where $\tilde{C}_{\omega_j}(B) = \sum_{p=0}^{+\infty} \tilde{C}_{\omega_j,p} B^p$ and $\tilde{C}_{\omega_j,p} = -\sum_{q=p+1}^{+\infty} C_q e^{i\omega_j(p-q)}$ such that $\{\tilde{C}_{\omega_j}(B) \varepsilon_t\}$ is a (complex) covariance stationary process. For each frequency, we have

$$C(B) S_{\omega_j} \varepsilon_t = C(e^{i\omega_j}) S_{\omega_j} \varepsilon_t + \tilde{C}_{\omega_j}(B) \varepsilon_t \quad (12)$$

Let denote r_j the rank of $C(e^{i\omega_j})$. If $r_j < n$, there exists a full rank $r_j \times n$ complex matrix $\alpha_{j,\perp}$ such that $\alpha_{j,\perp} C(e^{i\omega_j}) = 0$. This means that $\alpha_{j,\perp} C(B) S_{\omega_j} \varepsilon_t = \alpha_{j,\perp} \Delta_{x,-j}(B) x_t = \alpha_{j,\perp} \tilde{C}_{\omega_j}(B) \varepsilon_t$ is a covariance-stationary complex process when $\{\Delta_{x,-j}(B) x_t\}$ is an integrated process of order one at frequency ω_j . In this case, we say that this process is cointegrated at frequency ω_j (Hylleberg *et al.* (1991), Cubadda (2001), Gregoir (1999a)). This property means that the sources of persistent seasonal movements in the n -dimensional process are of a smaller dimension, here $(n - r_j)$. More precisely, $C(e^{i\omega_j})$ being of rank r_j , there exist two full rank $(n - r_j) \times n$ complex matrices such that $C(e^{i\omega_j}) = \alpha_j' \beta_j$ where α_j' stand for the transpose conjugate matrix of α_j . Equation (12) can then be rewritten as follows:

$$C(B) S_{\omega_j} \varepsilon_t = \alpha_j' \beta_j S_{\omega_j} \varepsilon_t + \tilde{C}_{\omega_j}(B) \varepsilon_t$$

$\{\beta_j' S_{\omega_j} \varepsilon_t\}$ is a $(n - r_j)$ -dimensional complex process that generates the persistent seasonal movements in the n -dimensional process. We then derive a very similar decomposition to that obtained in the case of a univariate process. We decompose $\{y_t, t \geq 0\}$ into the sum of a component without persistent seasonal components $\{y_t^*, t \geq 0\}$ and a component purely persistent at the seasonal frequencies $\{y_t^s, t \geq 0\}$ with the following definition

$$y_t = y_t^* + y_t^s$$

with

$$y_t^* = \mu_{v_0} + \mu_{x,\omega_0} + \phi_0 C(B) S_0 \varepsilon_t + \sum_{j=1}^k \phi_j \tilde{C}_{\omega_j}(B) \varepsilon_t$$

and

$$y_t^s = \sum_{j=1}^l \mu_{v_j} e^{i v_j t} + \sum_{j=1}^k \mu_{x,\omega_j} e^{-i \omega_j t} + \sum_{j=1}^k \phi_j \alpha_j' \beta_j S_{\omega_j} \varepsilon_t$$

Nevertheless, the filter Ψ that is implicitly applied on the process $\{y_t - d_t\}$ to get the “seasonally adjusted” process $\{y_t^* - \mu_{v_0}\}$ is more complicated due to the possible non invertibility of $C(B)$ at the frequencies ω_j . To derive this filter, we need additional assumptions and a representation theorem. Under the assumptions that $\forall \omega_j \in \Omega_x, rank C(e^{i\omega_j}) =$

r_j and that $\det C(B) = \prod_{j=0}^k \delta_{\omega_j}(B)^{r_j} d(B)$ where $d(B)$ is a polynomial whose roots have modulus strictly larger than one, a representation theorem in Gregoir (1999a) allows us to claim there exist a polynomial matrix $D(B)$ such that $D(0) = I_n$ and $\sum_{p=0}^{+\infty} \text{Tr}(D_p D'_p) < +\infty$, a set of $(k+1)$ couples of full rank $(r_j \times n)$ complex matrices⁶ (γ_j, θ_j) such that $\beta_j \gamma'_j = 0$ and

$$D(B) \Delta_x(B) x_t = \sum_{m=0}^k \gamma'_m \theta_m \Delta_{x,-m}(B) x_t + \varepsilon_t$$

It follows that

$$\alpha'_j \beta_j S_{\omega_j} \varepsilon_t = \alpha'_j \beta_j \left[D(B) \Delta_{x,-j}(B) - \sum_{m=0, m \neq j}^k \gamma'_m \theta_m \Delta_{x,-(m,j)}(B) \right] x_t$$

where

$$\Delta_{x,-(m,j)}(B) = \prod_{l=0, l \neq m, l \neq j}^k \delta_{\omega_l}(B)$$

The multivariate filter takes then the following form

$$\Psi(B) = I_n - \sum_{j=1}^k \frac{1}{\Delta_{x,-j}(e^{i\omega_j})} \alpha'_j \beta_j \left[D(B) \Delta_{x,-j}(B) - \sum_{m=0, m \neq j}^k \gamma'_m \theta_m \Delta_{x,-(m,j)}(B) \right]$$

It is therefore equal to I_n in 0 and close to I_n in its neighborhood. At each frequency ω_j , $j = 1, \dots, k$, it is not equal to 0 but not necessarily of full rank.

4.3 Drawbacks and advantages

The above general framework relies on a weak linear descriptive approach. It only assumes that the polynomial associated to the Wold representation of the process appropriately differenced satisfies Condition 9 or 10. Without any consensus on the relevant structural models appropriate to describe the sources of seasonal patterns in economic data, this corresponds to the usual pragmatic approach followed by most of the time series analysts. Its principle relies on the algebraic property that any process integrated at various frequencies can be described as a sum of processes integrated at each frequency with the same innovation process. This decomposition is always valid for univariate and multivariate processes, which is not true when we consider a decomposition into non-correlated components. The basic idea consists then of removing from the data the pure seasonal random walk components present in each of this integrated process. It is possible to remove alternative components that include this pure random walk component, but this would entail a larger distortion between the raw and the seasonally adjusted data. The adjustment does not introduce zeros in the spectral density of the seasonally adjusted series at the seasonal frequencies.

The implicit filter associated to this treatment is unidirectional. Its gain is always equal to 1 at frequency 0. It nevertheless implies that it induces a possible phase shift

⁶Where if (γ_j, θ_j) is associated to the frequency ω_j then $(\overline{\gamma_j}, \overline{\theta_j})$ is associated to the frequency $2\pi - \omega_j$.

that may affect the reading of the significant medium term changes in the data. Some examples available from the author indicate that this shift may be small in the neighborhood of 0. The decomposition basically relies on a Beveridge-Nelson decomposition at each frequency, we may wonder if it is possible to consider extensions of these filters that are symmetric. Proietti and Harvey (2000) introduce such an extension to the Beveridge-Nelson filter at frequency 0 when the persistent and transitory components are orthogonal but the decomposition associated to this filter is not always possible.

5 Practical implementation in a SARIMA framework

In the univariate case, implementing the above approach supposes that first we are able to take a decision about the adapted description of the time series under study in terms of seasonal persistent components (are they deterministic or stochastic ?), second we are able to provide users with an estimate of these persistent seasonal components.

5.1 Tests for seasonal integration in possible presence of deterministic terms at the same frequencies

From a practical point of view, statisticians are more inclined to consider models in which the seasonal pattern can change. The use of processes integrated at the seasonal frequencies allows for such features. A strategy of interest would then test for the presence of seasonal unit root in the presence of deterministic terms oscillating at the same frequencies.

The development of tests for seasonal unit roots has followed that of tests for unit roots at frequency 0. At this frequency, several procedures have been developed that have progressively allowed for some improvements in the size and power properties of the unit root tests. The most well-known procedure is that introduced by Dickey and Fuller (1979,1981). It relies on an autoregressive representation of the covariance stationary process that can also be read as a semi-parametric approximation (Said and Dickey (1984)). Inference in this set-up is sensitive to the presence of deterministic terms. Phillips and Perron (1987) adopt a non-parametric approach in a moving average representation of the covariance stationary process. Schmidt and Phillips (1992) looking for a test statistic whose asymptotic distribution is not sensitive to the presence of deterministic terms develop LM type tests. This allows for some improvements of the power properties. Elliott, Rothenberg and Stock(1997) relying on the Neyman-Pearson Lemma and King (1981) work on point optimal testing develop a near efficient test strategy whose asymptotic power curve is close to the asymptotic envelope. At last, Ng and Perron (2002) in this last framework improve the selection procedure of the specification of the autoregressive approximation used in the computation of a nuisance-parameter free test statistic. A parallel development can be observed for test for seasonal unit roots, but two main strategies have been developed.

A first approach deals with the problem of testing simultaneously for the presence of all the unit roots that are associated to the seasonal difference operator equal to $1 - L^s$ (when the seasonal length is s). This corresponds to a test for unit roots at frequency 0 and all the seasonal ones. Dickey, Hasza and Fuller (1984) extend the Dickey-Fuller approach. Breitung and Franses (1998) extend the Phillips-Perron approach and Tanaka (1996) presents an extension of Elliott, Rothenberg and Stock approach. Notwithstanding,

Taylor (2003) shows that this approach may lead the applied researcher to accept the seasonal unit root null hypothesis whereas the data generating process admits a unit root at the zero but not seasonal frequencies.

In a second approach, the statistician tests for the presence of each seasonal unit root (or couple of conjugate unit roots as we work with real time series). The procedures and asymptotic distributions depend on the frequencies. More precisely, asymptotic results for $\omega \in \{0, \pi\}$ and for $\omega \in]0, \pi[$ differ. At the quarterly or monthly frequencies, Hylleberg, Engle, Granger and Yoo (1991), Beaulieu and Miron (1993) and Ghysels, Lee and Noh (1994) use a test strategy that relies on the augmented Dickey-Fuller regression and consider Student and Fisher test statistics. Rodrigues (2002) extends the Schmidt and Phillips approach. Gregoir (2001) in a framework which draws a parallel between the test for seasonal unit roots and that for unit root at 0 frequency extends the Elliott, Rothenberg and Stock approach. A simulation study shows that this near-efficient test outperforms in terms of power properties most of the already available test statistics. In practice from empirical studies, it seems that numerous time series can be parsimoniously described as integrated ones but at a subset of the eligible frequencies. The introduction of a deterministic oscillating component may be necessary at the remaining ones.

5.2 Test for cointegration at seasonal frequencies

As illustrated above, when working with multivariate time series, the filter we obtain has a form that depends on the rank of the matrices $C(e^{i\omega_j})_{j=0,\dots,k}$. At frequency 0, this rank condition is related to the possible existence of cointegration between the integrated of order one time series. Cointegration at frequency 0 was introduced by Granger (1983), Granger and Weiss (1988) and Engle and Granger (1987). This concept was successful and generated a large theoretical and applied literature. In the simplest case, it corresponds to situation in which there exist linear combinations of integrated of order one processes at frequency 0 that are covariance stationary. In an autoregressive representation, the data satisfies then a Vector Error Correction Model. Two approaches have been used to test for the number of independent cointegration relationship and to estimate them. A two-step approach is proposed by Engle and Granger (1987), Stock and Watson (1988), Gregoir and Laroque (1994) and Harris (1997). Maximum likelihood approach is analyzed by Johansen (1988, 1991). Phillips (1988, 1991) analyzes this property in a spectral approach.

The rank condition of the matrix $C(e^{i\omega_j})$ when $j \neq 0$ leads to the concept of cointegration at seasonal frequencies which was introduced by Engle, Granger and Hallman (1989) and Hylleberg, Engle, Granger and Yoo (1991). A representation theorem that gives the form of the autoregressive model that is satisfied by the data in these circumstances is in Gregoir (1999a). The same two approaches have been used to test for the number of independent cointegration relationship at frequency ω_j and to estimate them. Lee (1992), Johansen and Schaumburg (1998) and Cubadda et al. (2003) provide us with maximum likelihood procedures and Gregoir (1999b) with a two-step procedure based on principal components. Cubadda (1995) carries out also an analysis working on the spectral density.

5.3 Demodulation and long term persistent component

5.3.1 Univariate SARIMA process

The computation of an estimate of the persistent seasonal component at frequency ω mimics that of the trend in a standard Beveridge-Nelson decomposition. More precisely, it corresponds to the same computation but on the demodulated process at frequency ω . In the presentation of the idea, we assume that only one seasonal unit root is present in the data generating process of the complex time series under studies $\{x_t\}$ and that the polynomial associated to its Wold representation satisfies Condition 9:

$$\delta_\omega(B) x_t = c(B) \varepsilon_t$$

The spectral density of this process is infinite at frequency ω . Following the same lines of reasoning as those in preceding section, we can write

$$x_t = \mu_\omega e^{-i\omega t} + c(e^{i\omega}) S_\omega \varepsilon_t + \tilde{c}_\omega(B) \varepsilon_t$$

where $\{\tilde{c}_\omega(B) \varepsilon_t\}$ is a complex covariance stationary process and by definition $S_\omega \varepsilon_t = \sum_{\tau=1}^t \varepsilon_\tau e^{-i\omega(t-\tau)}$. The demodulated observable process at frequency ω is $\{e^{i\omega t} x_t\}$ which is such that

$$e^{i\omega t} x_t = \mu_\omega + c(e^{i\omega}) \sum_{\tau=1}^t e^{i\omega\tau} \varepsilon_\tau + e^{i\omega t} \tilde{c}_\omega(B) \varepsilon_t$$

whose spectral density is infinite at frequency 0. We denote $\mathcal{I}_t$ the information available at date t , we assume given by $\{x_\tau, \tau \leq t\}$. The weak linear expectation of the demodulated process at date $t+h$ conditional on this information set is such that for $t \geq 0$:

$$\begin{aligned} E_t e^{i\omega(t+h)} x_{t+h} &= E e^{i\omega(t+h)} x_{t+h} | \mathcal{I}_t \\ &= \mu_\omega + c(e^{i\omega}) \sum_{\tau=1}^t e^{i\omega\tau} \varepsilon_\tau + \\ &\quad + E_t \left[c(e^{i\omega}) \sum_{\tau=t+1}^{t+h} e^{i\omega\tau} \varepsilon_\tau + e^{i\omega(t+h)} \tilde{c}_\omega(B) \varepsilon_{t+h} \right] \\ &= \mu_\omega + c(e^{i\omega}) \sum_{\tau=1}^t e^{i\omega\tau} \varepsilon_\tau + E_t [e^{i\omega(t+h)} \tilde{c}_\omega(B) \varepsilon_{t+h}] \end{aligned}$$

and therefore

$$\lim_{h \rightarrow +\infty} E_t e^{i\omega(t+h)} x_{t+h} = \mu_\omega + c(e^{i\omega}) \sum_{\tau=1}^t e^{i\omega\tau} \varepsilon_\tau$$

If we now notice that

$$\begin{aligned} e^{i\omega(t+h)} x_{t+h} &= e^{i\omega t} x_t + \sum_{\tau=1}^h (e^{i\omega(t+\tau)} x_{t+\tau} - e^{i\omega(t+\tau-1)} x_{t+\tau-1}) \\ &= e^{i\omega t} x_t + \sum_{\tau=1}^h e^{i\omega(t+\tau)} \delta_\omega(B) x_{t+\tau} \end{aligned}$$

we conclude that

$$\mu_\omega + c(e^{i\omega}) \sum_{\tau=1}^t e^{i\omega\tau} \varepsilon_\tau = e^{i\omega t} x_t + \sum_{\tau=1}^{+\infty} e^{i\omega(t+\tau)} E_t \delta_\omega(B) x_{t+\tau}$$

and a demodulation of this process at frequency $-\omega$ gives an estimate of the persistent component at frequency ω based on $\{x_\tau, \tau \leq t\}$:

$$\mu_\omega e^{-i\omega t} + c(e^{i\omega}) S_\omega \varepsilon_t = x_t + \sum_{\tau=1}^{+\infty} e^{i\omega\tau} E_t \delta_\omega(B) x_{t+\tau} \quad (13)$$

The interpretation of the Beveridge-Nelson decomposition in terms of extraction signal is presented in Watson (1986). His arguments remain valid.

In a general case, when the process is integrated of order 1 at various frequencies, we have

$$\Delta_x(B) x_t = c(B) \varepsilon_t$$

and from (9)

$$x_t = \sum_{j=0}^k \phi_j (\mu_\omega e^{-i\omega t} + c(e^{i\omega}) S_\omega \varepsilon_t + \tilde{c}_\omega(B) \varepsilon_t)$$

The above argument remains valid if we replace $\{x_t\}$ by $\{\Delta_{x,-j} x_t\}$ and equation (13) takes the form

$$\mu_\omega e^{-i\omega t} + c(e^{i\omega}) S_\omega \varepsilon_t = \Delta_{x,-j} x_t + \sum_{\tau=1}^{+\infty} e^{i\omega\tau} E_t \Delta_x(B) x_{t+\tau}$$

which gives an estimate of the persistent seasonal components to remove from the original time series equal to

$$\sum_{j=1}^k \frac{1}{\Delta_{x,-j}(e^{i\omega_j})} \left(\Delta_{x,-j} x_t + \sum_{\tau=1}^{+\infty} e^{i\omega\tau} E_t \Delta_x(B) x_{t+\tau} \right)$$

We now consider the situation in which $\{x_t\}$ satisfies a SARIMA representation. The computation of the Beveridge-Nelson decomposition at frequency 0 under the assumption that the process satisfies an ARIMA representation is given by Newbold (1990). We simply extend this method. We start from the DGP

$$\Phi(B) \Phi_s(B^s) \Delta_x(B) x_t = \Theta(B) \Theta_s(B^s) \varepsilon_t$$

where $d^0 \Phi = P$, $d^0 \Phi_s = P_s$, $d^0 \Theta = Q$, $d^0 \Theta_s = Q_s$ and all the roots of these polynomials have a modulus strictly larger than 1. We consider the $(P + sP_s + Q + sQ_s) \times 1$ process

$$\begin{aligned} z_t &= \left(\Delta_x(B) x_t \quad \dots \quad \Delta_x(B) x_{t-(P+sP_s)} \quad \varepsilon_t \quad \dots \quad \varepsilon_{t-(Q+sQ_s)} \right)' \\ \zeta_t &= \left(\varepsilon_t \quad 0 \dots 0 \quad \varepsilon_t \quad 0 \dots 0 \right)' \end{aligned}$$

and the usual $(P + sP_s + Q + sQ_s) \times (P + sP_s + Q + sQ_s)$ companion matrix $\mathcal{C}$ associated to this process such that

$$z_t = \mathcal{C} z_{t-1} + \zeta_t$$

Its weak linear forecast at horizon τ is given by

$$E_t z_{t+\tau} = \mathcal{C}^\tau z_t$$

whence if we denote K a $(P + sP_s + Q + sQ_s) \times 1$ vector whose first component is equal to 1 and the other ones to 0:

$$\begin{aligned} K' \sum_{\tau=1}^{+\infty} e^{i\omega\tau} E_t \Delta_x(B) x_{t+\tau} &= K' \left(\sum_{\tau=1}^{+\infty} e^{i\omega\tau} \mathcal{C}^\tau \right) z_t \\ &= K' (I_{P+sP_s+Q+sQ_s} - e^{i\omega} \mathcal{C})^{-1} e^{i\omega} \mathcal{C} z_t \end{aligned}$$

where the inverse of $(I_{P+sP_s+Q+sQ_s} - e^{i\omega} \mathcal{C})$ exists by the assumption made on the position of the roots of the polynomials $\Phi(B)$, $\Phi_s(B^s)$, $\Theta(B)$ and $\Theta_s(B^s)$. It follows that when the process satisfies a SARIMA model, the seasonal component is given by

$$x_t^s = \sum_{j=1}^k \frac{\Delta_{x,-j} x_t + K' (I_{P+sP_s+Q+sQ_s} - e^{i\omega} \mathcal{C})^{-1} e^{i\omega} \mathcal{C} z_t}{\Delta_{x,-j}(e^{i\omega_j})}$$

and the seasonally adjusted time series is given by

$$x_t^* = x_t - \sum_{j=1}^k \frac{\Delta_{x,-j} x_t + K' (I_{P+sP_s+Q+sQ_s} - e^{i\omega} \mathcal{C})^{-1} e^{i\omega} \mathcal{C} z_t}{\Delta_{x,-j}(e^{i\omega_j})}$$

5.3.2 Multivariate seasonally cointegrated processes

When working with a multivariate time series, the first part of the above reasoning can be used as it relies only on linear properties. For sake of simplicity, we keep to a n -dimensional process $\{x_t\}$ whose components are all integrated of order 1 at the same frequencies and to the description of its data generating process given in section 4.2. We have

$$\Delta_x(B) x_t = C(B) \varepsilon_t$$

and under condition 10, we obtain that an estimate of the persistent seasonal components to remove from the original time series is given by

$$\sum_{j=1}^k \frac{1}{\Delta_{x,-j}(e^{i\omega_j})} \left(\Delta_{x,-j} x_t + \sum_{\tau=1}^{+\infty} e^{i\omega\tau} E_t \Delta_x(B) x_{t+\tau} \right)$$

The difficulty is in the computation of the forecast and the vector model satisfied by the process $\{\Delta_x(B) x_t\}$. In absence of cointegration at any frequency, the above approach applies with a large companion matrix obtained in considering the corresponding $n(P + sP_s + Q + sQ_s) \times 1$ process z_t . In presence of cointegration, a constrained companion matrix must be considered, we refer to Gregoir (2004). The use of this constrained companion matrix is necessary to ensure the summability of $(e^{i\omega\tau} \mathcal{C}^\tau)_\tau$.

6 Conclusion

The approach we followed here is very natural in a weak linear set-up. It is valid under relatively weak conditions. As any model based method, one limit of this approach is related to the selection of the model as the implicit treatment or filter depends on the model. The model selection heavily depends on the presence of the unit roots and therefore on our ability to test for this null hypothesis. When considering multivariate time series, we know that the statistical decision taken about the presence or not of these roots in each of its components affects inference in the other parameters of the multivariate model. A change in the model implies a change in the filter. We nevertheless can notice that the estimate at date t is derived from the observation of the released raw data and the current model under the assumption that it is to remain unchanged in the future. In other words, in case of univariate time series if the data generating process may be well approximated by a slowly changing model with unpredictable exogenous changes, the seasonal component can still be computed.

The practical implementation of this method is relatively flexible, various estimation methods can be used. In particular regarding the problem of the treatment of outliers, we can use robust estimation method. As the seasonal component is construct from the estimates and the current set of information, Monte-Carlo or bootstrap methods such as that presented by Thombs and Schucany (1990) can be used to compute confidence interval of the seasonal adjusted time series.

The main issue is the control of the phase shift. The use of symmetric filter allows for an absence of shift in phase in the middle of the sample but the problem remains at the end and beginning of the sample. In the above set-up, the filter is always asymmetric and the phase shift can be non zero in the middle of the sample. Its magnitude depends on the particular model we consider. In order to control the properties of the statistical treatment, it must be recommended to compute the phase shift distortion in the selected model.

7 Reference

This list is not exhaustive.

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