

Declustering Peaks Over Threshold in Financial Time Series

Declusterizzazione di eccessi oltre la soglia in serie finanziarie

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Riassunto: Questo lavoro fornisce un metodo alternativo per la rappresentazione degli eccessi oltre una soglia in una serie finanziaria. Si utilizza una procedura di identificazione dei massimi di un *cluster* di valori estremi tramite l'informazione congiunta della serie dei rendimenti e della sua volatilità, qui misurata con modelli GARCH. L'efficienza degli stimatori di massima verosimiglianza viene migliorata modificando il concetto di *cluster* di valori estremi. Nella procedura di estrapolazione occorre considerare l'indice estremo delle serie finanziarie. Un esempio su dati dell'indice tedesco DAX30 illustrerà, brevemente, la metodologia.

Keywords: Dependence, Extremal index, Pareto model, Value at Risk

1. Introduction

During the last decade, market-risk measures have become increasingly relevant amongst statisticians. Market risk might be analyzed by financial time series data, and, in this context, Value at Risk (VaR) plays a prominent role. Essentially, the VaR of a portfolio is the amount of a potential loss which can be encountered once in a fixed period of time, with a (fixed) low probability.

For the estimate of this measure, several approaches have been proposed (Gouriéroux and Jasiak, 2001), with the main difference resting on the issue of considering or neglecting the information provided by the (unobserved) volatility of the market. In this paper we focus only into the estimate of unconditional VaR, i.e. we wish to make inference across different regimes of volatility. The interest in estimating unconditional VaR is of great concern for those institutions which have to hold shares for long period of time, with market conditions evolving in time.

In this work, estimating the unconditional VaR consists in extrapolating a quantile of an extreme value distribution, but we have to take into account that returns are far to be independent. Asymptotic theory of extreme value models holds for weakly dependent data, provided that extremes do not tend to cluster each other too often. In practice, for financial data, this assumption seems reasonable. Asymptotic likelihood results can be obtained by weakening the dependence, i.e. by *declustering* data (Coles, 2001). This implies the following strategy: a) Find a rule to define independent clusters; b) Extract the maximum from each cluster; c) Fit Pareto model to cluster maxima.

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2. Pareto model for independent processes

We start by considering a sequence of identical and independent (*iid*) random variables $\tilde{X}_1, \tilde{X}_2, \dots$ having common marginal distribution function F . It looks rather natural to consider as *extremes* only those elements of a sequence which are bigger than a specified high quantile u of F . Thus, we move our attention to terms $\tilde{X}_i > u$, where $\tilde{X}_i$ is a generic observation above u . For $y > 0$ the conditional probability $\Pr\{\tilde{X} > u + y \mid \tilde{X} > u\} = \{1 - F(u + y)\} / \{1 - F(u)\}$ describes the probabilistic behaviour of the distance between $\tilde{X}$ and u , given that a value bigger than u is observed. Limit results that do not involve F are well known (Coles, 2001). Assume that $\tilde{M}_{i,j} = \max\{\tilde{X}_i, \dots, \tilde{X}_j\}$ has one of the three-type extremal distribution G , which are Gumbel, Fréchet or Weibull. Then, for u big enough, $\{X - u \mid X > u\}$ follows a Generalized Pareto Distribution (GPD), with distribution function H_X , i.e.

$$\Pr(X - u \leq y \mid X > u) \approx H_X(y) = 1 - \left(1 + \frac{\xi y}{\beta}\right)^{-1/\xi}. \quad (1)$$

$H_X(y)$ is defined on a set such that $\{y : y > 0\}$ and $\{1 + \xi y/\beta > 0\}$. β is a location-scale parameter, while ξ is the shape parameter. For $\xi < 0$, H_X has an upper bound of $u - \beta/\xi$ and for $\xi > 0$ there is no upper bound. The case $\xi = 0$ has to be interpreted as the limit $\xi \rightarrow 0$ of expression (1) and H_X is exponential with parameter $1/\beta$. Notice that, for financial time series, we usually have $\xi > 0$.

3. Declustering dependent processes and extremal index

In current practice, financial observations are dependent and we need to trace back the independence through declustering data. However, it is common to assume that log-returns, defined in Section 4, are stationary. Asymptotic theory for the maxima of a stationary sequences $M_{i,j} = \max(X_i, \dots, X_j)$, with the same marginal distribution of $\{\tilde{X}_t\}$, is related through a parameter $\theta \in (0, 1]$, called the extremal index, as follows

$$H_{\tilde{X}} = \{H_X\}^\theta.$$

Cluster of extremal observations are related to the extremal index θ , being θ equal, in the limit, to the mean size of a cluster of exceedances. Therefore, natural estimators for θ simply need cluster of extreme observations to be identified.

Traditionally, clusters of extremal observations are considered as independent when they are far enough apart. In the classical runs estimator (RE), studied by Smith and Weissman (1994), two clusters are separated when, after observing a $X_i > u$, we have m consecutive observations below u ; it turns out that, when u is fixed, the RE strongly depends on runs length m . The RE comes from the conditional probability $\theta(u, m) = \Pr\{X_2 \leq u, \dots, X_m \leq u \mid X_1 > u\}$, with $\theta(u, 1) = 1$. For suitable sequences $m_n \rightarrow \infty$ and u_n such that $F(u_n) \rightarrow 1$, we have the convergence

$$\lim_{n \rightarrow \infty} \{\theta(u_n, m_n)\} = \theta.$$

Traditional runs method has recently been criticized by Laurini and Tawn (2002) as, in distinguish clusters, it does not account for the trajectory of the volatility process. Laurini and Tawn (2002) generalize runs method by adding a low threshold in the volatility

process $\{\sigma_t\}$, that, when exceeded, suggests the end of an extreme event. We discuss the strategy adopted by Laurini and Tawn (2002) for estimating θ , referred to as the two-thresholds (TT) method. The volatility-threshold approximation of θ is given by

$$\theta_\sigma(u, m, c) = \Pr\left(M_{2,m}^X \leq u \text{ or } \cup_{i=2}^m \{M_{2,i}^X \leq u, \sigma_i \leq c\} \mid X_1 > u\right). \quad (2)$$

We can express the event in probability (2) into mutually exclusive events as follows

$$\theta_\sigma(u, m, c) = \Pr\{M_{2,m}^X \leq u, L_{2,m}^\sigma > c \mid X_1 > u\} + \sum_{i=2}^m \Pr\{V_{i,c,u} \mid X_1 > u\}$$

where $i = 2, \dots, m$, $V_{i,c,u} = \{M_{2,i}^X \leq u, L_{2,i-1}^\sigma > c, \sigma_i \leq c\}$, and $L_{i,j}^\sigma = \min(\sigma_i, \dots, \sigma_j)$. Thus, $\theta_\sigma(u, m, c) \geq \theta(u, m)$ for all fixed values of u and m , with clusters terminating either if the runs condition on the values of the process is satisfied or if the conditional variance of the process becomes small enough between two exceedances of the $\{X_t\}$ process.

Properties of $\theta_\sigma(u, m, c)$ are discussed in Laurini (2003) and Laurini and Tawn (2002). Notice that if $c_n \rightarrow 0$ as $n \rightarrow \infty$ at a suitable rate then $\theta_\sigma(u_n, m_n, c_n) = \theta(u_n, m_n) \rightarrow \theta$ as $n \rightarrow \infty$.

4. Empirical Analysis of DAX30 Index

We work on negated DAX30 Index log-returns $r_t = \log(P_t) - \log(P_{t-1})$, where P_t is the price at day t , sampled at daily frequency from Dec-68 to Jan-03. To identify independent clusters we need an estimate $\hat{\sigma}_t$ of log-returns volatility. Volatility $\hat{\sigma}_t$ is estimated with simple GARCH(1, 1) model, which, however, capture the volatility clustering of our data.

The issue of threshold choice u_0 (the smallest value for which asymptotics hold) is addressed exploiting the mean residual life plot, which, for DAX30 log-returns, was linear from the 90-th quantile of the marginal distribution of the r_t process, i.e. $u_0 \approx 1.26$. Afterwards, through QQ and PP plots, we compare the goodness of GPD fit for both TT and RE declustering scheme; see Figure 1. An estimate of the unconditional VaR is given through return level, with confidence interval based on profile log-likelihood. The estimate of the return level is obtained by solving for $H(q_p) = 1/p$, defined in (1), as follows

$$\hat{q}_p = u_0 + \frac{\hat{\beta}}{\hat{\xi}} \left(\left(\hat{\theta} \frac{1-p}{N_u/n} \right)^{-\hat{\xi}} - 1 \right)$$

where $\hat{\beta}$ and $\hat{\xi}$ are, respectively, the ML estimates for location-scale and shape parameters; p represents the level of probability required (usually set as $1 - p = 0.05$); N_u/n is the proportion of threshold exceedances observed; and finally $\hat{\theta}$ is the extremal index estimate obtained either with RE or TT.

For our data, Figure 2 shows the return levels which indicate a value that would be exceeded, on average, once every 5-year of daily data. The better performance of TT method comes by inspecting Figure 1 and Figure 2. In the former, a better goodness of fit is shown for TT method, which has more data available for the fitting. In the latter, RE method seems to under-estimate the VaR and, compared to TT, shows wider confidence

intervals, which imply a higher level of uncertainty. Notice that ML estimates of RE and TT methods suffer from some bias. Bias is reduced in the TT method as more cluster maxima are available for fitting. However, further research is needed to inspect the size of such bias.

Figure 1: Diagnostics for GPD fit in cluster maxima for DAX30 through PP plot and QQ plot for cluster maxima obtained by declustering data with TT and RE methods

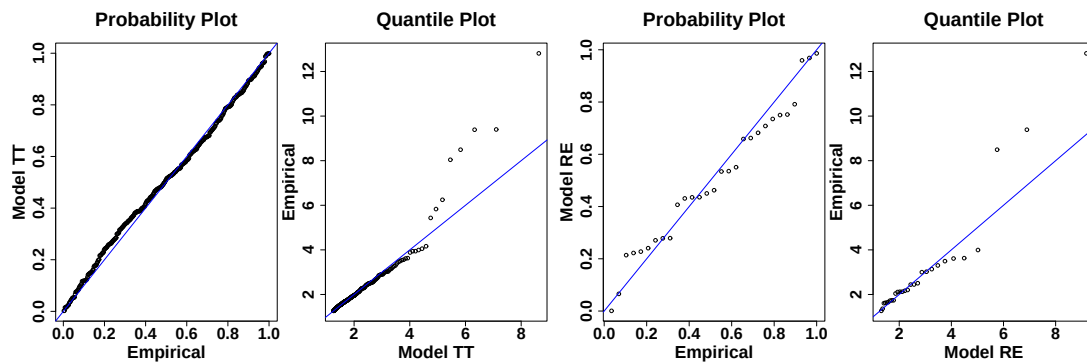
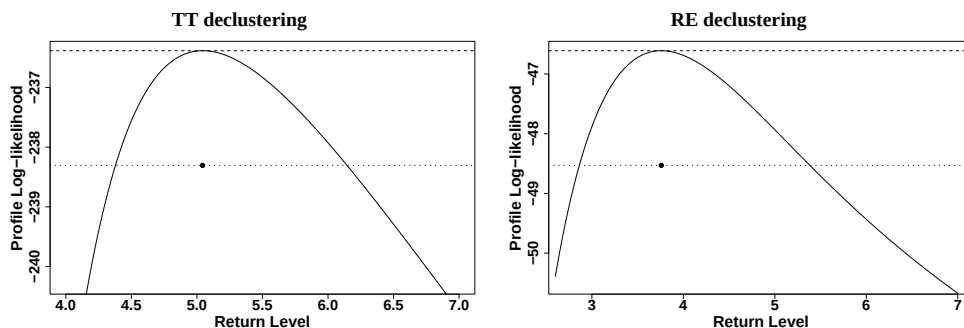


Figure 2: 5-year return level estimate for DAX30 Index based on cluster maxima analysis with TT and RE declustering scheme. In both panels dot is the maximum likelihood estimate and dotted line gives 95% confidence intervals



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