

Choosing between predictable and stochastic volatility

Test per volatilità stocastica

Valentina Corradi¹

Department of Economics
Queen Mary, University of London
v.corradi@qmul.ac.uk

Walter Distaso²

Department of Economics
University of Exeter
w.distaso@ex.ac.uk

Riassunto: Il presente paper propone un test per distinguere tra i casi in cui la volatilità di un asset finanziario è una funzione dell'asset stesso (e quindi è un processo di Markov ed è predicibile data la sua realizzazione nel passato), ed il caso in cui è una diffusione guidata da un moto Browniano non perfettamente correlato con il moto Browniano che guida l'asset.

La statistica, basata sulla differenza di due stimatori della volatilità, entrambi consistenti sotto l'ipotesi nulla, converge sotto l'ipotesi nulla ad una normale mista e diverge sotto l'alternativa.

Keywords: Ito diffusion models, stochastic volatility, realized volatility, mixed normality.

1. Motivation

The recent empirical literature on modeling the behaviour of prices of financial assets using continuous time models has focused on the use of multifactor models, where the variability of the process is driven by two or more (possibly, but not perfectly correlated) brownian motions.

The objective of this paper is to propose a testing procedure in order to distinguish between the cases in which the volatility of an asset is a function of the asset itself (and therefore the volatility process is Markov and predictable in terms of its own past), and the one in which it is a diffusion process driven by a Brownian motion, which is not perfectly correlated with the Brownian motion driving the asset.

Broadly speaking, in the former case the stochastic differential equation governing the behaviour of the asset is a one-dimensional Ito diffusion process, while the latter case coincides with the one of stochastic volatility, or in general with the case of more than just one factor-model.

The intuition behind the result is that realized volatility converges to integrated volatility regardless of whether volatility is predictable or stochastic. Then we compare realized volatility with a nonparametric estimator of integrated volatility, which is consistent in the case in which volatility is a measurable function of the underlying asset.

The crucial step, in order to have a nondegenerate asymptotic distribution for the proposed test statistic, is to calculate the realized volatility over a less fine grid of evaluation datapoints than the one used to get the nonparametric estimator. In fact, since the test statistic is essentially an Hausman type test, it turns out that if we use the same grid when

¹We are grateful to Nour Meddahi and Peter Phillips for very helpful comments and suggestions. The authors gratefully acknowledge financial support from the ESRC, grant code R000230006.

²Address for correspondence: Walter Distaso, University of Exeter, Department of Economics, Streatham Court, Exeter EX4 4PU, UK.

calculating the nonparametric estimator and the realized volatility, the two estimators are equally efficient and therefore the limiting distribution of the test statistic is degenerate.

We show that the difference between the two estimators converges to a mixed normal under the null hypothesis of predictable volatility and instead diverges under the alternative of stochastic volatility.

2. Setup

Consider the following continuous time process indicating the (log of the) price of a financial asset or the exchange rate between two currencies, defined by

$$dX_t = \sqrt{\sigma_t^2} dW_{1,t} \quad (1)$$

and

$$\sigma_t^2 = \sigma^2(X_t) \quad (2)$$

or

$$\begin{aligned} dX_t &= \sigma_t dW_{1,t} \\ \sigma_t^2 &= g(f_t) \\ df_t &= \sigma_1(f_t) dW_{2,t}, \end{aligned} \quad (3)$$

where f_t is an unobservable state variable driven by a Brownian motion, $W_{2,t}$, not perfectly correlated with the Brownian motion driving X_t .

We want to test the following set of hypothesis

$$H_0 : \sigma_t^2 = \sigma^2(X_t), \quad \text{a.s.}$$

versus the alternative

$$H_A : \sigma_t^2 = g(f_t), \quad \text{a.s.,}$$

where $\forall \omega \in \Omega^+, \left| \int_0^1 (g(f_s) - \sigma^2(X_s)) ds \right| \neq 0$ and $\Pr(\Omega^+) = 1$.

Consider the following test statistic

$$Z_{n,m} = \sup_{r \in (0,1)} Z_{n,m,r},$$

where

$$Z_{n,m,r} = \frac{\sqrt{m}}{n} \sum_{i=1}^{[(n-1)r]} S_n^2(X_{i/n}) - \sqrt{m} \sum_{j=1}^{[(m-1)r]} RV_{m,r},$$

and $[a]$ stands for the integer part of a . Here

$$S_n^2(X_{i/n}) = \frac{\sum_{j=1}^{n-1} 1_{\{|X_{j/n} - X_{i/n}| < \xi_n\}}^n (X_{(j+1)/n} - X_{j/n})^2}{\sum_{j=1}^{n-1} 1_{\{|X_{j/n} - X_{i/n}| < \xi_n\}}} \quad (4)$$

and

$$RV_{m,r} = \sum_{j=1}^{[(m-1)r]} (X_{(j+1)/m} - X_{j/m})^2. \quad (5)$$

$S_n^2(X_{i/n})$ is a nonparametric estimator of the volatility process evaluated at $X_{i/n}$; Florens-Zmirou (1993) has established consistency and the asymptotic distribution of a scaled version of (4) when the variance process follows (2). $RV_{m,r}$ is realized volatility, proposed as a proxy for volatility by Barndorff-Nielsen and Shephard (2002), Andersen *et al.* (2001) and Andersen *et al.* (2003). Realized volatility is a “model free” estimator of the quadratic variation of the process defined in (1), and is consistent under (2) and (3).

In the sequel, we assume that we have n recorded observations on a fixed time span, set for convenience equal to 1 (e.g. 1440 1-minute observations over a day), and that $m = o(n^{1-\varepsilon})$.

We will assume the following to hold.

Assumption A1:

- 1) $\sigma(\cdot)$ and $\sigma_1(\cdot)$ satisfy local Lipschitz and growth conditions. Therefore, for any compact subsets M and J of the range of the processes respectively under the null and the alternative hypothesis, there exist constants K_1^M , K_2^M , K_1^J and K_2^J such that, $\forall(x, y) \in M$ and $\forall(p, q) \in J$,

$$|\sigma(x) - \sigma(y)| \leq K_1^M |x - y|, \quad |\sigma_1(p) - \sigma_1(q)| \leq K_1^J |p - q|$$

and

$$|\sigma(x)| \leq K_2^M (1 + |x|), \quad |\sigma_1(p)| \leq K_2^J (1 + |p|).$$

- 2) $g(\cdot)$ is a Lipschitz-continuous function.

Since we are studying the diffusion processes over a fixed time span, we do not need to impose more demanding assumptions, such as stationarity and ergodicity.

Now we establish the limiting distribution of the proposed test statistic.

Proposition 1. *Let A1 hold. If, as $n \rightarrow \infty$, $n\xi_n \rightarrow \infty$ and $n\xi_n^2 \rightarrow 0$, then:*

- (i) under H_0 ,

$$Z_{n,m} \xrightarrow{d} \sup_{r \in (0,1)} Z_r,$$

where

$$Z_r = \text{MN} \left(0, 4 \int_{-\infty}^{\infty} \sigma^4(a) \bar{L}(r, a) da \right),$$

$\text{MN}(\cdot, \cdot)$ refers to a mixed normal distribution and

$\bar{L}(r, a) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\sigma^2(a)} \frac{1}{\varepsilon} \int_0^r 1_{[a, a+\varepsilon]}(X_u) \sigma^2(X_u) du = \frac{1}{\sigma^2(a)} L(r, a)$ is the standardized local time (see e.g. Florens-Zmirou, 1993; Bandi and Phillips, 2003);

- (ii) under H_A ,

$$\Pr(|Z_{n,m}| > \varepsilon(m\xi_n)^{1/2}) \rightarrow 1, \quad \text{for some } \varepsilon > 0.$$

Remark 1. As the covariance kernel is an operator (covariances across all r), we need to find valid asymptotic critical values either via the bootstrap or subsampling.

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