

Quantile Estimation of Minimum and Maximum Between the Components of a Bivariate Weibull Distribution

Stima dei Quantili del Minimo e del Massimo delle Componenti di un Modello di Weibull Bivariato

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Riassunto: Nei sistemi complessi formati da due o più elementi, un aspetto di particolare interesse è la determinazione della durata dell'intero sistema. Ci si propone di determinare degli opportuni stimatori di tale quantità nella duplice ipotesi che gli elementi del sistema siano collegati in "serie" o in "parallelo", situazioni coincidenti rispettivamente con la stima del valor "minimo", minor durata delle componenti, e della stima del "massimo", maggior durata delle componenti. Lo studio è condotto considerando una distribuzione dei tempi al guasto secondo la legge di Weibull nel caso di componenti correlate ed ottenendo le distribuzioni delle stime dei punti percentili mediante simulazione numerica.

Keywords: Bivariate Weibull, Quantile estimation, Method of moments estimation

1. Introduction

In literature many models have been proposed for the study of complex systems. The model proposed by Marshall and Olkin (1967) is based on the assumption that a two components system is distributed as a bivariate Exponential (BVE) model with the interesting condition that these two components are not independent. In reliability field an alternative distribution to the Exponential, is the Weibull law. In fact if the random variable X is exponentially distributed, $X^{1/\beta}$ has a Weibull distribution (Johnson, Kotz (1970)). In this paper we will study a bivariate Weibull distribution (BVW) model with non independent components in the same structure proposed by Marshall and Olkin. The aim of this paper is to provide an alternative estimator for the α -quantile with respect to those proposed in literature (Kaigh, Lachenbruch (1982) and Rukhin, Strawderman (1982)). In particular we consider the estimator of the α -quantile for the "minimum" life time and the "maximum" life time of the BVW model.

The method of moments is used for estimating the parameters of the BVW distribution and a Monte Carlo study is also provided in order to determine the statistical properties of the estimators proposed.

2. The bivariate Weibull model

The BVW model is studied under the assumption that a "fatal shock" could break the system that is to say that when a fatal shock comes, one of the components will fail. Thus if the random variable $U=(U_1, U_2)$ is BVW distributed, we can write

$$\bar{F}_U(u_1, u_2) = \Pr\{U_1 > u_1, U_2 > u_2\} = \exp\{-\lambda_1 u_1^{\beta_1} - \lambda_2 u_2^{\beta_2} - \lambda_{12} \max(u_1^{\beta_1}, u_2^{\beta_2})\} \quad (1)$$

where $\lambda_1, \lambda_2, \lambda_{12} > 0$ are the scalar parameters, than they represent the failure rate, and $\beta_1, \beta_2 > 0$ are the shape parameters. The two marginal distributions (for $i=1,2$) are Weibull distributed with the following laws:

$$\bar{F}_{U_i}(u_i) = \Pr\{U_i > u_i\} = \exp\{-(\lambda_i + \lambda_{12})u_i^{\beta_i}\} = \exp\{-\lambda_i^* u_i^{\beta_i}\} \text{ with } \lambda_i^* = \lambda_i + \lambda_{12} \quad (2)$$

and they can also be independent in case that λ_{12} is null, on the other case the bivariate distribution presents a singular region representing the case that both the components fail at the same time.

We can define the distribution law of the “minimum” life of the components of a bivariate Weibull model $V = \min(U_1, U_2)$ as:

$$\bar{F}_V(v) = \bar{F}_U(v, v) = \Pr\{(U_1 > v) \cap (U_2 > v)\} = \exp\{-\lambda_1 v^{\beta_1} - \lambda_2 v^{\beta_2} - \lambda_{12} \max(v^{\beta_1}, v^{\beta_2})\}. \quad (3)$$

In case $\beta_1 \neq \beta_2$ the distribution law is more complex because the value of $\max(v^{\beta_1}, v^{\beta_2})$ is equal to $v^{\max(\beta_1, \beta_2)}$ if $v > 1$ or is equal to $v^{\min(\beta_1, \beta_2)}$ if $v < 1$ and than the equation (3) is not directly invertible. In case $\beta_1 = \beta_2 = \beta$, situation that frequently we can observe in applicative fields, an easier distribution is obtained:

$$\bar{F}_V(v) = \exp\{-\lambda_1 v^\beta - \lambda_2 v^\beta - \lambda_{12} v^\beta\} = \exp\{-\lambda v^\beta\} \quad \text{with } \lambda = \lambda_1 + \lambda_2 + \lambda_{12} \quad (4)$$

that is an univariate Weibull distribution.

The law of the “maximum” life of the components of a BVW model $W = \max(U_1, U_2)$ is

$$\begin{aligned} \bar{F}_W(w) &= 1 - F_U(w, w) = 1 - \Pr\{(U_1 < w) \cap (U_2 < w)\} = \\ &= \exp\{-\lambda_1^* w^{\beta_1}\} + \exp\{-\lambda_2^* w^{\beta_2}\} - \exp\{-\lambda_1 w^{\beta_1} - \lambda_2 w^{\beta_2} - \lambda_{12} \max(w^{\beta_1}, w^{\beta_2})\}. \end{aligned} \quad (5)$$

In an analogous way if $\beta_1 \neq \beta_2$, we will observe the same difficulties we have observed in case of “minimum” life time, but if $\beta_1 = \beta_2 = \beta$ we obtain

$$\bar{F}_W(w) = \exp\{-\lambda_1^* w^\beta\} + \exp\{-\lambda_2^* w^\beta\} - \exp\{-\lambda w^\beta\} \quad \text{with } \lambda = \lambda_1 + \lambda_2 + \lambda_{12}. \quad (6)$$

3. Simulation experiments

The simulation experiments of this paper take into consideration the situation where the components of the system are similar that is to say when $\beta_1 = \beta_2 = \beta$. This situation permits to have some mathematical simplification.

The parameters of the BVW distribution has been estimated using the method of moments (Chiodini (1997)) that has been preferred to other estimation methods (e.g. ML - *Maximum Likelihood*, see Arnold (1968) and Castillo, Hadi (1995) p. 422) for its more simple mathematical form. The estimate of the α -percentile of the “minimum” of

the two marginals (v_α) in case of similar components is directly obtainable by the inverse function of the (4), after estimating the distribution parameters λ and β . We can observe that the estimate of the α -percentile of the “maximum” of the two marginals (w_α) is not directly obtainable from the inverse function of the (6), even if we assume the case where the marginals are similar. In this case, after introducing the method of moments in order to obtain the estimate of the distribution parameters λ_1^* , λ_2^* , λ , β , we have to introduce an approximate method, by means of the Taylor’s series, for solving the equation (6) and estimate the α -percentile.

A study on the properties of the estimates has been made with the Monte Carlo method in order to investigate different situations. For all this different sets of parameters ($n = 10, 20, 100$, $\beta = 2, 3$, $\rho = 0.25, 0.5$, $\lambda_1^* = 0.75, 1$, $\lambda_2^* = 1$, $\alpha = 5\%$), 1000 random samples have been generated. In Table 1 and in Table 2, that differs only for the shape parameter β , some of the most interesting cases studied are resumed. Bias, mean square error (Mse) and 5%, 50% (median) and 95% percentile of the estimates are presented.

We observe that in all the simulated cases the estimates are generally positively biased but quite strongly decreasing with the increase of the sample dimension. The same analysis can be made with regard to the mean square error. In all the cases the estimates of the distribution parameters and the α -percentile for the “minimum” and “maximum” seem to be quite robust with regard to the sample dimension. In fact the dimension of the 90% interval (the interval that contains the 90% of the estimates) decreases quite quickly while the sample dimension increases.

Table 1: Case study: $\lambda_1^*=0.75$, $\lambda_2^*=1$, $\lambda=1.4$, $\beta=2$, $\rho=0.25$ ($v_\alpha=1.4628$, $w_\alpha=2.0698$)

$n=10$	$\hat{\lambda}_1^*$	$\hat{\lambda}_2^*$	$\hat{\lambda}$	$\hat{\beta}$	$\hat{v}_\alpha$	$\hat{w}_\alpha$
Bias	0.0699	0.0721	0.0783	0.1710	-0.0157	-0.0356
Mse	0.1083	0.1574	0.2792	0.2779	0.0506	0.1077
5% percentile	0.4012	0.5390	0.7651	1.5312	1.1011	1.5316
50% percentile	0.7631	1.0167	1.4180	2.0903	1.4254	2.0098
95% percentile	1.4463	1.7800	2.4435	3.0691	1.8400	2.5984
90% interval	1.0451	1.241	1.6784	1.5379	0.7389	1.0668
$n=20$	$\hat{\lambda}_1^*$	$\hat{\lambda}_2^*$	$\hat{\lambda}$	$\hat{\beta}$	$\hat{v}_\alpha$	$\hat{w}_\alpha$
Bias	0.0327	0.0384	0.0417	0.0645	-0.0034	-0.0133
Mse	0.0388	0.0650	0.1222	0.0929	0.0293	0.0547
5% percentile	0.5063	0.6688	0.9435	1.6186	1.1995	1.6952
50% percentile	0.7628	1.0125	1.4075	2.0500	1.4469	2.0435
95% percentile	1.1410	1.4966	2.0884	2.6063	1.7507	2.4616
90% interval	0.6347	0.8278	1.1449	0.9877	0.5512	0.7664
$n=100$	$\hat{\lambda}_1^*$	$\hat{\lambda}_2^*$	$\hat{\lambda}$	$\hat{\beta}$	$\hat{v}_\alpha$	$\hat{w}_\alpha$
Bias	0.0095	0.0073	0.0405	-0.0111	-0.0089	0.0110
Mse	0.0067	0.0126	0.0344	0.0155	0.0080	0.0112
5% percentile	0.6286	0.8404	1.1644	1.7911	1.3128	1.9069
50% percentile	0.7548	1.0022	1.4315	1.9842	1.4513	2.0799
95% percentile	0.8999	1.1995	1.7592	2.2028	1.6093	2.2598
90% interval	0.2713	0.3591	0.5948	0.4117	0.2965	0.3529

Table 2: Case study: $\lambda_1^*=0.75$, $\lambda_2^*=1$, $\lambda=1.4$, $\beta=3$, $\rho=0.25$ ($v_\alpha=1.2886$, $w_\alpha=1.6241$)

$n=10$	$\hat{\lambda}_1^*$	$\hat{\lambda}_2^*$	$\hat{\lambda}$	$\hat{\beta}$	$\hat{v}_\alpha$	$\hat{w}_\alpha$
Bias	0.0875	0.0722	0.1104	0.1580	-0.0148	-0.0095
Mse	0.1056	0.1520	0.2676	0.4532	0.0167	0.0283
5% percentile	0.4442	0.5811	0.8157	2.2694	1.0734	1.3393
50% percentile	0.7771	1.0029	1.4242	3.0759	1.2695	1.6087
95% percentile	1.4506	1.7649	2.4183	4.3898	1.4932	1.8959
90% interval	1.0064	1.1838	1.6026	2.1204	0.4198	0.5566
$n=20$	$\hat{\lambda}_1^*$	$\hat{\lambda}_2^*$	$\hat{\lambda}$	$\hat{\beta}$	$\hat{v}_\alpha$	$\hat{w}_\alpha$
Bias	0.0381	0.0414	0.0525	0.0611	-0.0048	-0.0029
Mse	0.0385	0.0674	0.1252	0.2153	0.0090	0.0154
5% percentile	0.5109	0.6815	0.9336	2.4097	1.1273	1.4213
50% percentile	0.7630	1.0043	1.4203	3.0050	1.2818	1.6215
95% percentile	1.1392	1.4696	2.0712	3.9129	1.4366	1.8305
90% interval	0.6283	0.7881	1.1376	1.5032	0.3093	0.4092
$n=100$	$\hat{\lambda}_1^*$	$\hat{\lambda}_2^*$	$\hat{\lambda}$	$\hat{\beta}$	$\hat{v}_\alpha$	$\hat{w}_\alpha$
Bias	0.0101	0.0125	0.0379	-0.0121	-0.0056	0.0028
Mse	0.0067	0.0115	0.0330	0.0371	0.0026	0.0029
5% percentile	0.6356	0.8465	1.1706	2.6918	1.1996	1.5400
50% percentile	0.7521	1.0088	1.4241	2.9776	1.2823	1.6246
95% percentile	0.8982	1.1987	1.7362	3.3199	1.3646	1.7189
90% interval	0.2626	0.3522	0.5656	0.6281	0.165	0.1789

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