

Asymptotic Laws of Minimum Dissimilarity Estimates of a Position Parameter

Leggi Asintotiche di Stime di Minima Dissomiglianza per un Parametro di Posizione

Federico Bassetti ¹

Dipartimento di Matematica "F.Casorati", Università degli Studi di Pavia.
bassetti@dimat.unipv.it

Riassunto: In questa breve nota ci occupiamo del problema di determinare la legge asintotica di stime di minima dissomiglianza per un parametro di posizione. In primo luogo mostriamo come questo problema può essere ricondotto al calcolo della legge del tempo di soggiorno "generalizzato" di un ponte browniano. Quindi otteniamo il risultato desiderato grazie ad un'estensione della formula di Feynman–Kac.

Keywords: Asymptotic laws, Feynman–Kac formula, minimum dissimilarity estimates.

1. Introduction and formulation of the problem

Given any probability distribution function A on $\mathbb{R}$, set

$$A_\mu(x) := A((x - \mu)/\sigma_0) \quad (x \in \mathbb{R}),$$

μ being an unknown position parameter and σ_0 a fixed scale parameter. We deal with the problem of estimating μ on the basis of independent observations $\tilde{x}_1, \tilde{x}_2, \dots$, identically distributed according to A_μ . More precisely, we consider the empirical probability distribution function $y \mapsto E_n(\tilde{x}(n); y) := \frac{1}{n} \sum_{k=1}^n \mathbb{I}_{[\tilde{x}_k, +\infty)}(y)$ ($y \in \mathbb{R}$), where $\tilde{x}(n) := (\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n)$ and $\mathbb{I}_E$ stands for the indicator of E , and define a *minimum dissimilarity* estimate $\tilde{\mu}_n$ for μ any minimizer of the *Gini dissimilarity index*

$$\int_{\mathbb{R}} |E_n - A_\mu| dy.$$

We have been motivated to study this kind of estimates because, in spite of the existence of more efficient procedures (such as maximum likelihood estimators), they generally surpass their competitors from the point of view of robustness, when robustness is meant, for example, in the sense of the *bias distortion curves* of Donoho and Liu (1988). We proceed to seek for the asymptotic distribution of $\sqrt{n}(\tilde{\mu}_n - \mu)$, as n tends to infinity and, under a few suitable conditions, show that the limiting distribution is the same as the distribution of

$$\hat{h} := \operatorname{argmin}_{h \in \mathbb{R}} \int_{(0,1)} \left| \frac{\sigma_0 \mathbb{B}(y)}{a(A^{-1}(y))} - h \right| dy$$

when $\mathbb{B}$ stands for a standard *Brownian bridge* and a for a derivative of A . Hence, for each path B of $\mathbb{B}$, $\hat{h}$ can be viewed as a median of the distribution of the random variable

¹Indirizzo per corrispondenza: Dipartimento di Matematica, Università di Pavia, Via Ferrata 1, 27100 Pavia.

$y \mapsto \sigma_0 B(y)/(a(A^{-1}(y)))$ defined on the probability space $((0, 1), \mathcal{B}(0, 1), \lambda)$, λ being the restriction of the Lebesgue measure to $\mathcal{B}(0, 1)$. This implies that the value at ξ of the probability distribution function of $\hat{h}$ coincides with the probability of the event

$$\left\{ \int_{(0,1)} \mathbb{I}_{(-\infty, \xi a(A^{-1}(s))/\sigma_0]}(\mathbb{B}(s)) ds \geq 1/2 \right\}.$$

Thus, one can determine the characteristic function $\phi(\cdot; \xi)$ of $\tau(\mathbb{B}; \gamma) := \int_{(0,1)} \mathbb{I}_{(-\infty, \gamma(s)]}(\mathbb{B}(s)) ds$, with $\gamma(s) := \xi a(A^{-1}(s))/\sigma_0$, and then, from a well-known inversion formula, obtain

$$\text{Prob}\{\hat{h} > \xi\} = \frac{1}{2} - \frac{1}{\pi} \lim_{\epsilon \rightarrow 0^+, c \rightarrow +\infty} \int_{\epsilon}^c \text{Im} \left[\frac{e^{-it/2}}{t} \phi(t; \xi) \right] dt. \quad (1)$$

See, for example, Kawata (1972). In this brief note, we (a) summarize the arguments that can be employed to determine the characteristic function $\phi(\cdot)$ of $\tau(\mathbb{B}; \gamma)$, and (b) illustrate these arguments by two examples. It is worth mentioning that the probability distribution of $\hat{h}$ turns out to be symmetric. Complete proofs can be found in the technical report Bassetti and Regazzini (2003b), while more explanation of the statistical aspects are contained in Bassetti and Regazzini (2003a).

2. Solution of the problem.

The characteristic function ϕ can be determined by resorting to the celebrated *Feynman–Kac theorem*, which states that $(x, t) \mapsto \mathbb{E}[f(\mathbb{W}_1^{(x,t)}) \exp\{\int_t^1 k(\mathbb{W}_s^{(x,t)}, s) ds\}]$, under suitable regularity conditions on f and k , solves the well-defined Cauchy problem

$$\begin{cases} -\frac{\partial}{\partial t} u(x, t) = \frac{\partial^2}{\partial x^2} u(x, t) + k(x, t) u(x, t) & (x, t) \in \mathbb{R} \times [0, 1) \\ u(x, 1) = f(x), \end{cases} \quad (2)$$

$\mathbb{W}_s^{(x,t)}$ being a Brownian motion indexed by s on $[t, 1]$ and satisfying $\mathbb{W}_t^{(x,t)} = x$. One can try to solve the previous Cauchy problem with $f(x) = \mathbb{I}_{(-\epsilon, \epsilon)}(x)$ and $k(x, t) = -z \mathbb{I}_{(-\infty, \gamma(t)]}(x)$ to obtain

$$\mathcal{F}_{\epsilon}(x, t; z) := \mathbb{E} \left[\mathbb{I}_{(-\epsilon, \epsilon)}(\mathbb{W}_1^{(x,t)}) \exp\left\{-z \int_t^1 \mathbb{I}_{(-\infty, \gamma(s)]}(\mathbb{W}_s^{(x,t)}) ds\right\} \right]$$

for every nonnegative ϵ, t in $(0, 1)$ and z, x in $\mathbb{R}$. Then, by recalling that a Brownian bridge can be identified as a Brownian motion conditioned such that it takes the value 0 at 1, one obtains the characteristic function ϕ of $\tau(\mathbb{B}; \gamma)$ through a passage to the limit, viz.

$$\phi(t) = \lim_{\epsilon \rightarrow 0^+} \mathcal{F}_{\epsilon}(0, 0; it) / \mathbb{P}\{\mathbb{W}_1^{(0,0)} \in (-\epsilon, \epsilon)\} \quad (t \in \mathbb{R}^+).$$

In point of fact, if γ and $\dot{\gamma}$ belong to $C^0[0, 1] \cap C^1(0, 1)$ and to $L^p(0, 1)$, respectively, for some $p > 2$, a slight modification of the Feynman–Kac theorem yields that $w_0(x, y) := \sqrt{2\pi} \lim_{\epsilon \rightarrow 0^+} \mathcal{F}_{\epsilon}(x, 1 - 2y; z) / (2\epsilon)$ is the unique bounded “generalized” solution of (2) with $f(x) = \delta_0(x)$ and $k(x, t) = -z \mathbb{I}_{(-\infty, \gamma(t)]}(x)$. Now, by assuming that w_0 and $\partial_x w_0$

are continuous, from the theory of the heat potentials we obtain that such a solution can be expressed as a function of the vector function $(\nu_0^+(y, z), \nu_0^-(y, z))$, which is the unique continuous solution of the following system of Volterra integral equations

$$\left\{ \begin{array}{l} \frac{(1-e^{-2zy})}{2\sqrt{\pi y}} \exp\left\{-\frac{(r(y))^2}{4y}\right\} + \int_0^y \frac{1}{2\sqrt{\pi(y-\tau)}} \nu_0^+(\tau) \exp\left\{-\frac{(r(y)-r(\tau))^2}{4(y-\tau)}\right\} d\tau \\ = \int_0^y \frac{1}{2\sqrt{\pi(y-\tau)}} \nu_0^-(\tau) \exp\left\{-2z(y-\tau) - \frac{(r(y)-r(\tau))^2}{4(y-\tau)}\right\} d\tau \\ - \frac{(1-e^{-2zy})}{2\sqrt{\pi y^{3/2}}} r(y) \exp\left\{-\frac{(r(y))^2}{4y}\right\} = \nu_0^+(y) + \nu_0^-(y) \\ + \int_0^y \frac{r(y)-r(\tau)}{2\sqrt{\pi(y-\tau)^{3/2}}} \nu_0^+(\tau) \exp\left\{-\frac{(r(y)-r(\tau))^2}{4(y-\tau)}\right\} d\tau \\ - \int_0^y \frac{r(y)-r(\tau)}{2\sqrt{\pi(y-\tau)^{3/2}}} \nu_0^-(\tau) \exp\left\{-2z(y-\tau) - \frac{(r(y)-r(\tau))^2}{4(y-\tau)}\right\} d\tau \end{array} \right. \quad (3)$$

with $r(y) := \gamma(1 - 2y)$. At this stage, the characterisic function of $\tau(\mathbb{B}; \gamma)$ can be expressed as

$$\phi(t) = \left\{ \begin{array}{ll} 1 + \int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-2u}} \exp\left\{-\frac{\gamma(1-2u)^2}{2(1-2u)}\right\} \nu_0^+(u, -it) du & \text{if } \gamma(0) < 0 \\ e^{it} + \int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-2u}} \exp\left\{it(1-2u) - \frac{\gamma(1-2u)^2}{2(1-2u)}\right\} \nu_0^-(u, -it) du & \text{if } \gamma(0) \geq 0. \end{array} \right.$$

3. Examples

We will illustrate the application of the previous result to the evaluation of (1) with two examples which involve distinguished families of distributions depending on a position parameter. It should be noted that the expression of w_0 which appears in the examples could be also deduced from 2.1.5.7 in Borodin and Salminen (2002).

3.1 Consider a family of exponential distributions with unknown position parameter and fixed scale parameter, i.e. $A((x-\mu)/\sigma_0) := (1 - \exp(-(x-\mu)/\sigma_0))\mathbb{I}_{[0,+\infty)}((x-\mu)/\sigma_0)$. Then, $r(y) = 2\tilde{\xi}y$ ($0 \leq y \leq 1/2$, $\tilde{\xi} := \xi/\sigma_0 \in \mathbb{R}$). In this case, system (3) can be easily solved to obtain $\nu_0^-(y) = \frac{1}{y}e^{-y\tilde{\xi}^2-yz}I_1(z y)$, and $\nu_0^+(y) = -\nu_0^-(y)$, where I_1 denotes the modified Bessel function of the first kind. Then, for any $\xi > 0$, we have

$$\phi(t; \xi) = e^{it} \left\{ 1 + \int_0^1 \frac{1}{u\sqrt{1-u}} \exp\left(-\frac{u^2}{2(1-u)}\tilde{\xi}^2 - \frac{u}{2}\tilde{\xi}^2 - \frac{it}{2}u\right) I_1\left(-\frac{it}{2}u\right) du \right\}.$$

If in (1) we substitute this expression of ϕ , we obtain

$$\begin{aligned} Prob\{\hat{h} > \xi\} &= \frac{1}{2} - \frac{1}{\pi} \lim_{\epsilon \rightarrow 0^+, c \rightarrow +\infty} \int_{\epsilon}^c \frac{1}{t} \left\{ \sin(t/2) - \int_0^1 \frac{1}{u\sqrt{1-u}} \right. \\ &\quad \cdot \exp\left\{-\frac{1}{2}\tilde{\xi}^2(u + \frac{u^2}{1-u})\right\} \cos(t(1-u)/2) J_1\left(\frac{ut}{2}\right) du \Big\} dt \end{aligned}$$

J_1 being the Bessel function of the first kind. Hence, by inverting the order of integration, we obtain

$$\begin{aligned} p(\theta_0)\{\hat{h} \leq \xi\} &= 1 - \frac{2^{3/2}}{\pi} \int_0^1 \frac{\sqrt{x}}{(1+x)^2\sqrt{1-x}} \exp\left\{-\frac{1+x}{2(1-x)}\tilde{\xi}^2\right\} dx \\ &= 1 - \frac{1}{2\sqrt{\pi}} \Psi\left(\frac{3}{2}, \frac{1}{2}, 2\tilde{\xi}^2\right) e^{-4\tilde{\xi}^2} \end{aligned}$$

for every $\xi \geq 0$, Ψ being the confluent hypergeometric function of the second kind (the Tricomi function). This statement, combined with the symmetry of the law of $\hat{h}$, allows the previous results to be expressed as follows: *Let $\tilde{\mu}_n$ be the minimum dissimilarity estimate (of order 1) of μ based on the first n elements of a sequence $(\tilde{x}_n)_{n \geq 1}$ of independent and identically distributed random variables having common exponential distribution with unknown position parameter μ_0 and fixed scale parameter σ_0 . Then, the asymptotic distribution (as $n \rightarrow +\infty$) of $\sqrt{n}(\tilde{\mu}_n - \mu_0)$ is absolutely continuous with density function*

$$\xi \mapsto \frac{2^{3/2}}{\pi \sigma_0^2} |\xi| \int_0^1 \frac{\sqrt{x}}{(1+x)\sqrt{(1-x)^3}} \exp\left\{-\frac{1+x}{2\sigma_0^2(1-x)} \xi^2\right\} dx \quad (\xi \in \mathbb{R}).$$

Its moment of order $2k$ is

$$1 \cdot 3 \cdot \dots \cdot (2k-1) \sigma_0^{2k} / (k+1) \quad (k = 1, 2, \dots).$$

3.2 We now deal with the uniform model, i.e. $A(x) := (x - \mu)/\sigma_0 \mathbb{I}_{(\mu, \mu+\sigma_0)}(x) + \mathbb{I}_{[\mu+\sigma_0, +\infty)}$. Then, $r(y) \equiv \tilde{\xi}$. Following arguments which are almost the same as in the previous example, we can state the following result: *Let $\tilde{\mu}_n$ be the minimum dissimilarity estimate (of order 1) of μ based on the first n elements of a sequence $(\tilde{x}_n)_{n \geq 1}$ of independent and identically distributed random variables having common uniform distribution with unknown position parameter μ_0 and fixed scale parameter σ_0 . Then the asymptotic distribution function (as $n \rightarrow +\infty$) of $\sqrt{n}(\tilde{\mu}_n - \mu_0)$ is*

$$U(\xi) := \begin{cases} \frac{|\xi|}{\sigma_0 \sqrt{2\pi}} \Psi\left(2, \frac{3}{2}, \frac{2\xi^2}{\sigma_0^2}\right) e^{-\frac{4\xi^2}{\sigma_0^2}} & \text{if } \xi \leq 0 \\ 1 - \frac{\xi}{\sigma_0 \sqrt{2\pi}} \Psi\left(2, \frac{3}{2}, \frac{2\xi^2}{\sigma_0^2}\right) e^{-\frac{4\xi^2}{\sigma_0^2}} & \text{if } \xi \geq 0, \end{cases} \quad (4)$$

whose $2k$ -th moment can be expressed as

$$\sigma_0^{2k} \frac{1 \cdot 3 \cdot \dots \cdot (2k-1)}{8^k (k+1)} F\left(\frac{3}{2}, k; k+2, \frac{1}{2}\right) \quad (k = 1, 2, \dots),$$

F being the Gauss hypergeometric function.

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