

The moments of SETARMA models and their interpretation

I momenti del modello SETARMA e loro interpretazione

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Riassunto: La conoscenza dei momenti di un processo stocastico consente di indagare su alcune sue proprietà e di valutare le diverse caratteristiche che la struttura è in grado di catturare nella descrizione di fenomeni empirici. Scopo del presente lavoro è di derivare i momenti di una particolare classe di modelli non-lineari a soglia, i modelli SETARMA. Dopo aver brevemente introdotto il modello, vengono in seguito proposte le forme esatte dei momenti e viene descritto il loro andamento al variare di alcuni parametri.

Keywords: SETARMA models, Moments, Asymmetry, Kurtosis.

1. The SETARMA models

Since the introduction of the SETARMA model in Tong (1983, 1990) many questions about this class of models still remain unanswered. Among them, the present paper address the attention to the unconditional moments generation and the exact formula of the central moments up to order four is shown. The asymmetry and kurtosis coefficients of the SETARMA process are further presented in order to show some aspects which make the process under analysis very interesting even in empirical studies.

The SETARMA process (Tong, 1983) of order $(k; p_1, \dots, p_k; q_1, \dots, q_k)$ is defined as:

$$X_t = \sum_{i=1}^k \left[\phi_0^{(i)} + \sum_{j=1}^{p_i} \phi_j^{(i)} X_{t-j} + \sigma_i \left(e_t - \sum_{w=1}^{q_i} \theta_w^{(i)} e_{t-w} \right) \right] I(X_{t-d} \in R_i) \quad (1)$$

where $\sigma_i e_t \sim WN(0, \sigma_i^2)$, $R_i = [r_{i-1}, r_i)$ forms a partition of the real line such that $-\infty = r_0 < r_1 < r_2 < \dots < r_k = +\infty$ with r_i the threshold values, d is the threshold delay, p_i and q_i are non negative integers, $\phi_j^{(i)}$ and $\theta_w^{(i)}$ are unknown parameters, with $j = 1, 2, \dots, p_i$ and $w = 1, 2, \dots, q_i$, and $I(\cdot)$ is a Bernoulli random process.

Applying the results of Box and Jenkins (1976) to the local linear structure of the SETARMA model, Amendola et al. (2003) suggest to write model (1) in an alternative manner, which is limited to the two regime case ($k = 2$) in order to easily present some theoretical results:

$$X_t = \left[c_0^{(1)} + \sigma_1 \sum_{j=0}^{\infty} \psi_j^{(1)} B^j e_t \right] I_{t-d} + \left[c_0^{(2)} + \sigma_2 \sum_{j=0}^{\infty} \psi_j^{(2)} B^j e_t \right] (1 - I_{t-d}) \quad (2)$$

where

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- $I_{t-d} = I(X_{t-d} \in R_1)$ and $1 - I_{t-d} = I(X_{t-d} \in R_2)$
- $c_0^{(1)} = \frac{\phi_0^{(1)}}{1 - \sum_{j=1}^{p_1} \phi_j^{(1)}}$ and $c_0^{(2)} = \frac{\phi_0^{(2)}}{1 - \sum_{j=1}^{p_2} \phi_j^{(2)}}$ are the mean values of the two regimes;
- $\sum_{j=1}^{\infty} |\psi_j^{(i)}| < \infty$, for $i = 1, 2$, with $\psi_0 = 1$
- and the weights are:

$$\psi_j^{(i)} = \begin{cases} 1 & \text{when } j = 0 \\ \sum_{s=0}^{j-1} \psi_s^{(i)} \phi_{j-s}^{(i)} - \theta_j^{(i)} & \text{when } j \geq 1 \end{cases} \quad (3)$$

with $\phi_j^{(i)} = 0$, for $j > p_i$, and $\theta_j^{(i)} = 0$, for $j > q_i$.

2. The generation of moments

The study of moments of X_t has an important role when the distribution of the process need to be investigated. In this context the moments of model (2) are derived under the assumptions that the errors e_t are Gaussian and X_t is strictly stationary and ergodic. Starting from the following proposition, the general expression of the moments up to order four are derived as function of the lower order unconditional expected values of X_t .

Proposition 1 *Given $X_t \sim SETARMA(2; p_1, p_2; q_1, q_2)$, suppose that the joint process $\mathbf{X}_t = (X_t^{(1)}, X_t^{(2)}, I_{t-d})$ [with $X_t^{(i)} \sim ARMA(p_i, q_i)$ the distribution of regime i], is strictly stationary and ergodic and that each regime admits at least moments of order r , then the expected value of X_t^r is:*

$$E[X_t^r] = \mu_r^{(1)}p + \mu_r^{(2)}(1 - p) \quad (4)$$

where $r \in \mathbb{N}$, $\mu_r^{(i)} = E[(X_t^{(i)})^r]$, for $i = 1, 2$ and $p = E[I_{t-d}]$.

Proof. (see Amendola et al., 2003, Section 3). ◇

The result (4) allows to show that the variance of X_t is:

$$Var(X_t) = E[(X_t - \mu)^2] = pVar(X_t^{(1)}) + (1-p)Var(X_t^{(2)}) + p(1-p)[c_0^{(1)} - c_0^{(2)}]^2 \quad (5)$$

with $Var(X_t^{(i)}) = \sigma_i^2 \Psi_{(i)}^2$ and $\Psi_{(i)}^2 = \sum_{j=0}^{\infty} (\psi_j^{(i)})^2$, for $i = 1, 2$, which assumes a simpler form when the expected value of the process X_t is null and so $c_0^{(1)} = c_0^{(2)} = 0$.

The expected value (4) further allows to obtain the third and fourth central moments which can be used to evaluate if the SETARMA model is able to catch the asymmetry and the kurtosis of data, as shown in the following two corollaries.

Corollary 1 *Under the hypothesis of Proposition 1, the third central unconditional moment of X_t is given as:*

$$E[(X_t - \mu)^3] = (c_0^{(1)} - c_0^{(2)}) \cdot [\bar{\mu}_{3I}(c_0^{(1)} - c_0^{(2)})^2 + 3Var(I_t)(\sigma_1^2 \Psi_1^2 - \sigma_2^2 \Psi_2^2)] \quad (6)$$

with $Var(I_t) = p(1 - p)$ and $\bar{\mu}_{3I} = p(1 - p)(1 - 2p)$.

Sketch of proof. Starting from:

$$E[(X_t - \mu)^3] = \mu_3 - 3\mu_2\mu + 4\mu^3 \quad \text{with} \quad \mu_r = E[(X_t)^r] \quad r = 1, 2, 3 \quad (7)$$

the assumption of Gaussianity of e_t and Proposition 1 imply that each term in (7) is:

$$\begin{aligned} \mu &= p \cdot c_0^{(1)} + (1-p) \cdot c_0^{(2)} \\ \mu_2 &= p[(c_0^{(1)})^2 + \sigma_1^2 \Psi_{(1)}^2] + (1-p)[(c_0^{(2)})^2 + \sigma_2^2 \Psi_{(2)}^2] \\ \mu_3 &= p[(c_0^{(1)})^3 + 3c_0^{(1)}\sigma_1^2 \Psi_{(1)}^2] + (1-p)[(c_0^{(2)})^3 + 3c_0^{(2)}\sigma_2^2 \Psi_{(2)}^2] \end{aligned}$$

that combined in (7), and after some algebra, give the result (6). $\diamond$

Corollary 2 Under the hypothesis on X_t given in Proposition 1, its fourth central unconditional moment is given by:

$$\begin{aligned} \bar{\mu}_4 = E[(X_t - \mu)^4] &= \bar{\mu}_{4I}(c_0^{(1)} - c_0^{(2)})^4 + 3 \left[p\sigma_1^4 (\Psi_{(1)}^2)^2 + (1-p)\sigma_2^4 (\Psi_{(2)}^2)^2 \right] + \\ &+ 6Var(I_t)(c_0^{(1)} - c_0^{(2)})^2 \left[p\sigma_1^2 \Psi_{(1)}^2 + (1-p)\sigma_2^2 \Psi_{(2)}^2 \right] \end{aligned} \quad (8)$$

where $\bar{\mu}_{4I} = E[(I_t - p)^4] = p(1-p)[1 - 3p(1-p)]$.

Sketch of proof. Given:

$$E[(X_t - \mu)^4] = \mu_4 - 4\mu_3\mu + 6\mu_2\mu^2 - 3\mu^4 \quad (9)$$

using the results in Proposition 1, μ_4 in (9) is:

$$\mu_4 = p \left[(c_0^{(1)})^4 + 3\sigma_1^4 (\Psi_{(1)}^2)^2 + 6(c_0^{(1)})^2 \sigma_1^2 \Psi_{(1)}^2 \right] + (1-p) \left[(c_0^{(2)})^4 + 3\sigma_2^4 (\Psi_{(2)}^2)^2 + 6(c_0^{(2)})^2 \sigma_2^2 \Psi_{(2)}^2 \right]$$

whereas μ , μ_2 and μ_3 are defined in Corollary 1. The substitution of the correspondent moments in (9), after some algebra, gives the following results:

$$\begin{aligned} E[(X_t - \mu)^4] &= (c_0^{(1)})^4 \bar{\mu}_{4I_t} + (c_0^{(2)})^4 \bar{\mu}_{4(1-I_t)} - 2Var(I_t)[1 - 3Var(I_t)]c_0^{(1)}c_0^{(2)} \cdot \\ &\cdot [2(c_0^{(1)})^2 - 3c_0^{(1)}c_0^{(2)} + 2(c_0^{(2)})^2] + 3[p\sigma_1^4 (\Psi_{(1)}^2)^2 + (1-p)\sigma_2^4 \cdot \\ &\cdot (\Psi_{(2)}^2)^2] + 6Var(I_t)(c_0^{(1)} - c_0^{(2)})^2 [p\sigma_1^2 \Psi_{(1)}^2 + (1-p)\sigma_2^2 \Psi_{(2)}^2] \end{aligned} \quad (10)$$

Observing that $\bar{\mu}_{4I_t} = Var(I_t)[1 - 3Var(I_t)]$ and summing and subtracting $(c_0^{(2)})^4 \bar{\mu}_{4I_t}$ in (10), the form (8) is obtained. $\diamond$

From (6) and (8) we can compute the skewness and kurtosis indexes:

$$\gamma_1 = E[(X_t - \mu)^3]/E[(X_t - \mu)^2]^{\frac{3}{2}} \quad \text{and} \quad \gamma_2 = E[(X_t - \mu)^4]/E[(X_t - \mu)^2]^2 - 3$$

from which follows that, given the assumption of Gaussian errors, when $c_0^{(1)} \neq c_0^{(2)}$ the SETARMA model is able to catch the asymmetry and the kurtosis of data. Further, when $c_0^{(1)} = c_0^{(2)}$ the skewness coefficient is $\gamma_1 = 0$ whereas, when the coefficients of both

regimes are equal, $\gamma_2 = 0$. This highlights the different role of the model coefficients in catching the aspects of data related to the form of the distribution.

It implies interesting consequences when the SETARMA models are applied in economic and financial domain, where the observed time series are affected by asymmetries, often related to business cycles and market inefficiencies, and high kurtosis.

In order to evaluate the behaviour of the skewness and kurtosis indexes, in Figure 1 and 2 are shown the plots of γ_1 and γ_2 for three different values of $p = 0.25, 0.50, 0.75$ of a SETARMA(2; 1,1; 1,1) model. In particular in Figure 1 are plotted the values assumed by γ_1 when $\phi_0^{(1)}$ and $\phi_0^{(2)}$ vary on the interval $[0, 0.99]$ whereas in Figure 2 is shown how γ_2 change with respect to different values of σ_1^2 and σ_2^2 .

Figure 1: *Skewness coefficient of the SETARMA(2; 1,1; 1,1) model for three different values of $p = 0.25, 0.50, 0.75$ and with $\phi_0^{(1)}, \phi_0^{(2)} \in [0, 0.99]$*

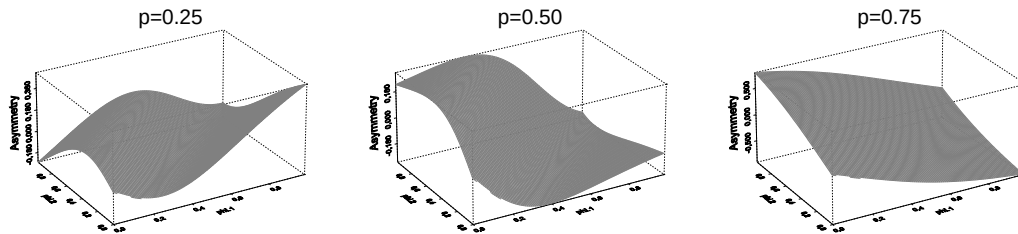
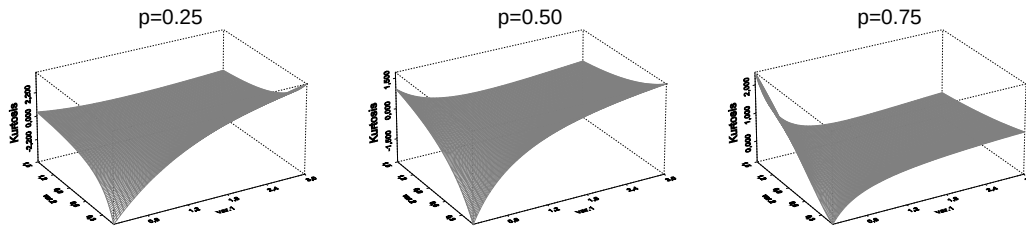


Figure 2: *Kurtosis coefficient of the SETARMA(2; 1,1; 1,1) model for three different values of $p = 0.25, 0.50, 0.75$ and with $\sigma_1^2 \in [0.1, 3]$ and $\sigma_2^2 \in [0.1, 1.5]$*



The choice of the plots is related to some results, not reported here, which highlight that the model intercepts have an heavier effect on γ_1 than on γ_2 whereas the kurtosis behaviour is mainly related to the variance of the regimes errors.

References

- Amendola, A., Niglio, M. and Vitale, C. (2003). The autocorrelation function in setarma models, Proceeding of: *Metodi nonlineari per la previsione di fenomeni economici*, Università Cà Foscari, Venezia, 20 October, forthcoming.
- Box, G. and Jenkins, G. (1976). *Time series analysis, forecasting and control*, Holden-Day.
- Tong, H. (1983). *Threshold models in nonlinear time series analysis*, Springer-Verlag.
- Tong, H. (1990). *Non-linear time series: A dynamical system approach*, Clarendon Press.