

Bayesian Estimators for the Parameters of the Power Law Process: a Closed Form Solution

Stimatori Bayesiani dei Parametri del Power Law Process: una Soluzione in Forma Chiusa

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Riassunto: In questo lavoro sono presentati alcuni nuovi stimatori dei parametri del Power Law Process. Questi stimatori, che sono stati espressamente formulati per applicazioni di interesse affidabilistico, hanno due proprietà che nessun altro metodo noto all'autore possiede contemporaneamente; infatti, essi oltre a permettere di incorporare nel processo di stima dell'affidabilità le informazioni anticipate dagli esperti del settore, presentano la peculiarità di produrre soluzioni in forma chiusa.

Keywords: Bayesian Methods, Reliability Estimation, Power Law Process

1. Introduction

A complex system is an item that can be restored to operating condition after a failure by repairing or replacing a part of it. In general the modelling of failure-repair pattern of these items constitutes a very hard task. Thus, some simplifying hypotheses are usually adopted to face this problem. In particular, when the main interest is in failure process, it is customary to assume that repairs are made instantaneously. Moreover, since many complex systems are usually repaired by replacing a very small part of them, it is often possible to assume that a repair returns the reliability of the system to the level it was just before the occurrence of the failure. A model that describes the failure-pattern under these hypotheses is the Non-Homogeneous Poisson Process (NHPP).

The most popular form of the Mean function of a NHPP is:

$$M(t) = \lambda \cdot t^b, \quad (1)$$

where λ is the mean number of arrivals at time $t=1$ and b is the shape parameter. In literature a NHPP with mean function (1) is referred to as Power Law Process (PLP) [Asher et al.].

In this paper some new bayesian estimators of the parameters of this model are presented. These estimators have two properties that no other method known to the author has both [Bar-Lev et al.], [Guida et al., 1988], [Guida et al., 1992], [Guida et al., 1996]. The first property is that they enable to incorporate technological (prior) information into the estimation process. The second one is that their use doesn't involve numerical methods.

2. Formulation of the estimators

Let $t_1 < t_2 < \dots < t_n$ be the first n times to failure of a NHPP with mean function (1). The likelihood function results in:

$$L(\lambda, b) = \lambda^n b^n P^{b-1} e^{-\lambda \cdot t_0^b}; \quad P = \prod_{i=1}^n t_i,$$

where the total time on test, t_0 , in failure truncated sampling is equal to t_n .

2.1. Prior distribution

The estimators proposed in this paper adopt the following joint prior distribution:

$$pdf(\lambda, b) = \frac{e^{-b \ln(T^v/\pi)} \cdot [\ln(T^v/\pi)]^{\nu-1} \cdot \lambda^{\nu-1} \cdot b^{\nu-2} \cdot e^{-w \cdot T^b \cdot \lambda}}{(w \cdot T^b)^{-\nu} \cdot (\nu-1) \cdot [\Gamma(\nu-1)]^2}. \quad (2)$$

The prior means of b , λ and $M(t)$ are the following:

$$\begin{aligned} E[b] &= \frac{\nu-1}{\ln(T^v/\pi)}; \quad E[\lambda] = \frac{\nu}{w} \cdot \left(\frac{E[b]}{\nu-1} \cdot \ln T + 1 \right)^{-(\nu-1)}; \\ E[\lambda|b] &= \frac{\nu}{w \cdot T^b}; \quad E[M(t)] = \frac{\nu}{w} \cdot \left[\frac{E[b]}{\nu-1} \cdot \ln(T/t) + 1 \right]^{-(\nu-1)}. \end{aligned} \quad (3)$$

2.2. Posterior distribution and point estimators

Combining the likelihood function and the prior distribution via the Bayes theorem, the following posterior distribution is obtained:

$$pdf(\lambda, b|\mathbf{x}) = \frac{P_p^b \cdot \lambda^{n_p-1} \cdot b^{n_p-2} \cdot e^{-(w \cdot T^b + t_0^b) \cdot \lambda}}{\Gamma(n_p) \cdot \int_0^\infty P_p^b \cdot b^{n_p-2} \cdot (w \cdot T^b + t_0^b)^{-n_p} \cdot db}; \quad \begin{aligned} n_p &= \nu + n \\ P_p &= \pi \cdot P \end{aligned}$$

that in general cannot be expressed in closed form [Huang et al.].

Nevertheless, by setting $T = t_0$ in prior (2), the integral can be solved analytically and the following posterior distribution is obtained.

$$pdf(\lambda, b | \mathbf{x}) = \frac{e^{-b \ln(t_0^{n_p}/P_p)} \cdot \left[\ln(t_0^{n_p}/P_p) \right]^{n_p-1} \cdot \lambda^{n_p-1} \cdot b^{n_p-2} \cdot e^{-w_p \cdot t_0^b \cdot \lambda}}{(w_p \cdot t_0^b)^{-n_p} \cdot (n_p-1) \cdot [\Gamma(n_p-1)]^2},$$

where $w_p = w + 1$.

This distribution gives closed form (posterior mean) point estimators of b , λ and $M(t)$:

$$E[b | \mathbf{x}] = \frac{n_p - 1}{\ln(t_0^{n_p}/P_p)}; \quad E[\lambda | \mathbf{x}] = \frac{n_p}{w_p} \cdot \left(\frac{E[b | \mathbf{x}]}{n_p - 1} \cdot \ln t_0 + 1 \right)^{-(n_p-1)}$$

$$E[\lambda | b, \mathbf{x}] = \frac{n_p}{w_p \cdot t_0^b}; \quad E[M(t) | \mathbf{x}] = \frac{n_p}{w_p} \cdot \left[\frac{E[b | \mathbf{x}]}{n_p - 1} \cdot \ln(t_0/t) + 1 \right]^{-(n_p-1)}.$$

2.3. Calibration of the prior

In order to calibrate the prior distribution the hyperparameters γ, π, T and w have to be used. This task can be facilitated observing that the posterior means of b , λ and $M(t_0)$ (i.e. $M(t)$ at time $t = t_0$) can be also expressed as follow:

$$E[b | \mathbf{x}] = \frac{n_p - 1}{n_p - 2} \cdot \left\{ \frac{\nu - 1}{(\nu - 1) + (n - 1)} \cdot \frac{1}{E[b]} + \frac{n - 1}{(\nu - 1) + (n - 1)} \cdot \frac{1}{E_N[b | \mathbf{x}]} \right\}^{-1}$$

$$E[\lambda | b, \mathbf{x}] = \frac{w}{w + 1} \cdot E[\lambda | b] + \frac{1}{w + 1} \cdot E_N[\lambda | b, \mathbf{x}] \quad , (4)$$

$$E[M(t_0) | \mathbf{x}] = \frac{n_p}{w_p} = \frac{w}{w + 1} \cdot E[M(t_0)] + \frac{1}{w + 1} \cdot E_N[M(t_0) | \mathbf{x}]$$

where:

$$E_N[b | \mathbf{x}] = \frac{n - 1}{\ln(t_0^n/P)}; \quad E_N[\lambda | b, \mathbf{x}] = \frac{n}{t_0^b}; \quad E_N[M(t_0) | \mathbf{x}] = n \quad (5)$$

are the estimators obtained by adopting the noninformative prior:

$$f(\lambda, b) \propto \frac{1}{\lambda \cdot b^2},$$

formulated using argument in [Box et al.], and $E[b]$, $E[\lambda|b]$, $E[M(t_o)]$ are the prior means, obtained by setting $T = t_o$ in equation (3). Equation (4) shows that posterior estimates are weighted averages of noninformative estimates (5) and prior estimates (3), with weighting factors that depend on v and w .

We finally note that the closed form solution is obtained by setting the hyperparameter T to t_o . So the analyst can use only γ, π and w to calibrate the prior. For instance, he could use these hyperparameters to set two prior means and one of the weighting factors previously considered ($E[b]$, $E[\lambda]$ and $(v-1)/(n+v-2)$ or $E[b]$, $E[M(t_o)]$ and $1/(w+1)$).

3. Conclusions

The bayesian estimators presented in this paper have two characteristics that make them very appealing. The first one is that they use a prior that enables to incorporate, in a natural and direct way, technological information into the reliability estimation process. The second one is that they can be expressed in closed form. This last property is important for at least two reasons. In fact it eliminates the computational gap of informative bayesian estimators with respect to other estimators of PLP parameters, like Maximum Likelihood estimators [Bain] or noninformative bayesian (5), that can be expressed in closed form. Moreover, the closed form gives evidence of prior information contribution to the estimation process, making the user more confident in the bayesian method.

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