

# Extreme Events and Risk Measures: a Comparison among Alternative Approaches

*Eventi Estremi e Misure di Rischio: un Confronto tra Approcci Alternativi*

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**Riassunto:** Nell'analisi del rischio (assicurativo, di credito o di mercato) assume particolare rilevanza poter disporre correntemente di misure sintetiche e facilmente calcolabili. Il Value-at-Risk (VaR) fornisce il valore della massima perdita che può verificarsi su un dato orizzonte temporale, per un certo livello di confidenza. La teoria dei valori estremi offre strumenti idonei alla modellazione di valori sulle code di una distribuzione, consentendo una stima di percentili elevati nella distribuzione delle perdite. Nel presente lavoro tale approccio viene messo a confronto con altri metodi utilizzati per la determinazione del VaR. L'attenzione viene, infine, posta sulla opportunità di tenere conto del fenomeno di *volatility clustering* tipico dell'analisi dei dati finanziari, coniugando in modo adeguato stime di volatilità condizionata con un approccio basato su valori che eccedono una determinata soglia (Peaks-over-Threshold model).

**Keywords:** Extreme Value Theory, Peaks over Threshold, Value at Risk.

## 1. Introduction

In this paper we focus on the extreme behaviour of financial data. We have to deal with extreme events when a risk takes values from the tails of its probability distribution. In the field of market risk management it is a great concern the day by day determination of the Value-at-Risk (VaR). VaR is a high quantile of the distribution of losses (for example the 95th or 99th percentile):  $Var_p = F^{-1}(p)$ , where  $F$  is the loss cumulative distribution function and  $p$  the selected probability level. The extreme value theory provides tools for estimating measures of tail risk. We consider a fully parametric model, based on the Generalized Pareto Distribution, which can be easily estimated by maximum likelihood. The structure of the paper is the following. First we introduce the Peaks-over-Threshold (POT) model, giving only some brief insights on the fundamentals of extreme value theory. Then we focus on the Value-at-Risk estimation making a comparison among alternative modeling techniques.

## 2. The extreme value approach

The extreme value theory is an useful tool in providing appropriate distributions to fit extreme events. Two main approaches are proposed in the literature: the Block Maxima models and the Peaks-over-Threshold models (POT). In this paper we focus on parametric POT models, based on the Generalized Pareto Distribution (GPD).

The cumulative distribution of *excess losses* ( $Y = X - u$ , where  $X$  is a random

variable and  $u$  is a given threshold) *i.e.* the conditional excess distribution function is defined as:

$$F_u(y) = P(X - u \leq y | X > u) = \frac{F(x) - F(u)}{1 - F(u)} \quad (1)$$

and can be approximated by a GPD (Embrechts *et al.* (1997)). The unconditional distribution can be written as:

$$F(x) = (1 - F(u))F_u(x) + F(u) = 1 - \frac{N_u}{n} \left(1 + \frac{\xi}{\beta}(x - u)\right)^{-1/\xi} \quad (2)$$

where  $F_u(x)$  is a GPD and  $F(u)$  is given by  $(n - N_u)/n$ ;  $n$  is the total number of observations,  $N_u$  the number of observations above the threshold  $u$ ,  $\xi$  and  $\beta$  are the parameters of the GPD. Inverting (2) for a given probability  $p$  we obtain:

$$VaR_p = u + \frac{\beta}{\xi} \left( \left( \frac{n}{N_u}(1 - p) \right)^{-\xi} - 1 \right) \quad (3)$$

By fitting a POT model we can estimate the distribution of the excess over the threshold. We can estimate the parameters  $\xi$  and  $\beta$  of the GPD by maximum likelihood method. Choice of the threshold  $u$  is the main issue to deal with:  $u$  too high results in too few exceedances and consequently high variance estimators. On the other hand,  $u$  too small provides biased estimators and the approximation to a GPD could not be feasible. It is possible to choose an asymptotically optimal threshold by a quantification of a bias versus variance trade-off. One suggestion which is of immediate use in practice is based on the linearity of the mean excess function  $e(u)$  for the GPD. From Kellezi and Gilli (2000) we know that for a random value  $X$  with a GPD distribution function  $G_{\xi,\beta}$  the mean excess function is:

$$e(u) = E(X - u | X > u) = \frac{\beta + \xi u}{1 - \xi}, \quad \beta + u\xi > 0, \quad \xi < 1 \quad (4)$$

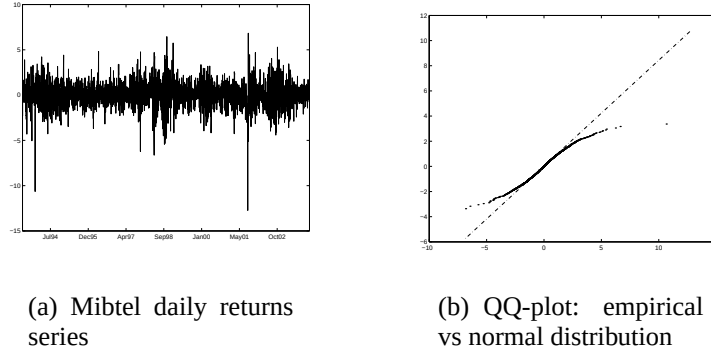
hence  $e(u)$  is linear. The (4) suggests a graphical approach for choosing  $u$ : choose  $u > 0$  such that  $e(x)$  is approximately linear for  $x \geq u$ . Using plots to compare resulting estimates across a variety of  $u$ -values, due to the usual presence of multiple choice of the threshold, is recommended.

### 3. Empirical results and conclusions

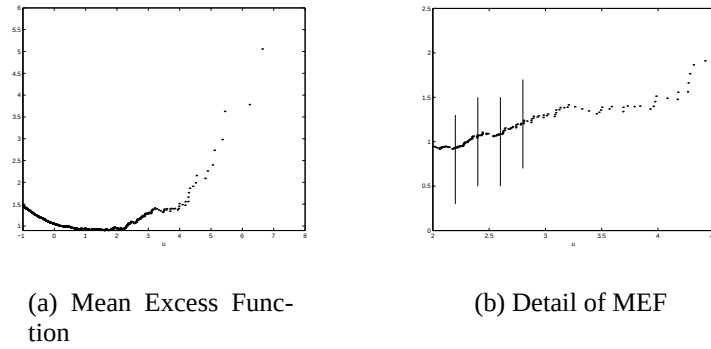
We compare the performance of the GPD model with different VaR modelling techniques, such as historical simulation, normal and GARCH(1,1) models. We consider a univariate approach, working on the series of daily negative returns (losses) of the Italian Mibtel index over a period of ten years (1993-2003). It should be noted that historical simulation, normal and GARCH models focus on returns under *normal* market conditions, while the POT approach is based on events occurring under *extreme* market conditions.

The empirical implementation of the POT method requires to face some particular issues: determining whether the series is fat-tailed, choosing the threshold and the estimate method for parameters. The mean excess function (4) allows us to establish the behavior of the tails: we choose  $u$  looking at the linear shape (with positive slope) of the graph,

**Figure 1:** Mibtel - sample period 20/07/93- 12/12/2003: returns series and QQ-plot vs normal distribution



**Figure 2:** Mean excess function and detail of MEF to choose the threshold  $u$



in particular (see Figure 2(a) and 2(b)) we consider  $u = 2.2$  (the number of observation exceeding  $u$  is equal to 138). The qq-plot graph makes us able to evaluate the goodness of fit of the series to a parametric model (see Figure 1(b) and 3(b)).

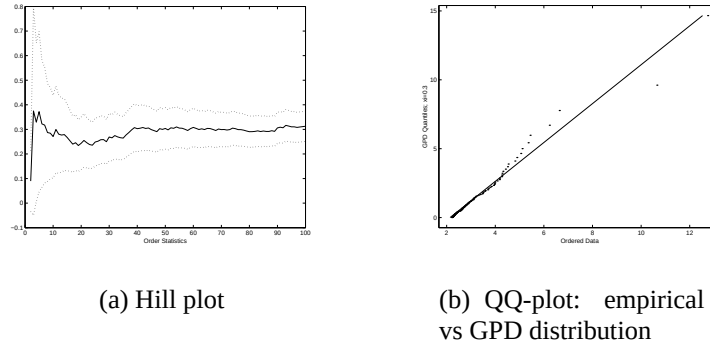
In Figure 3(a) we report the Hill graph, which plots the Hill estimator of  $\xi$ ,  $\hat{\xi} = \left(\frac{1}{k} \sum_{j=1}^k \ln X_{j,n} - \ln X_{k,n}\right)$ , versus the  $k$  upper order statistics. The Hill estimator is equal to the ML estimator, with the deterministic  $u$  replaced by the random threshold  $X_{k,n}$  where  $k = \text{card}\{i : X_{i,n} > u, i = 1, \dots, n\}$ . We select the area of the graph where the Hill estimator is stable. Finance empirical studies suggest as typical values  $\xi$  between 0.25 and 0.33. The ML estimates of the GPD parameters ( $u = 2.2$ ) are  $\xi = 0.30277$  (which corresponds to Hill's graph analysis) and  $\beta = 0.65543$ .

In Table 1 we report 95%, 99%, 99.5%, 99.9% Value-at-Risk estimates of five different VaR estimation methods. In the last row of the table the results obtained by a two-stage estimator which embeds a GARCH volatility estimation in the POT framework (see McNeil and Frey (2000)) are presented.

Consider that the EVT literature usually assumes i.i.d. data, which is not common in the analysis of financial time series. In order to take account of the presence of dependent data and improve the performance of the tail index estimator, we can estimate the tail of the conditional, rather than unconditional, distribution.

Due to the conditional nature of GARCH(1,1) and GARCH(1,1)-GPD approaches we obtain lower percentiles, as we can observe a low volatility period at the end of the sample considered (see Figure 1(a)).

**Figure 3:** Hill plot and QQ-plot vs GPD distribution



The performance of the different VaR estimation methods can be evaluated by comparing the estimates with the actual losses observed, in particular by computing (and testing) the number of VaR violations.

**Table 1:** VaR estimation for daily Mibtel losses (in percent)- one day horizons

| VaR approach          | 95th   | 99th   | 99.5th | 99.9th |
|-----------------------|--------|--------|--------|--------|
| Historical simulation | 2.2232 | 3.7025 | 4.3111 | 6.5459 |
| Normal model          | 2.3152 | 3.2792 | 3.6311 | 4.3489 |
| GPD model             | 2.2349 | 3.6161 | 4.4522 | 7.2256 |
| GARCH(1,1)            | 1.4606 | 2.0689 | 2.2908 | 2.7437 |
| GARCH(1,1)-GPD        | 1.4145 | 2.1476 | 2.557  | 3.8073 |

VaR approaches based on the assumption of normal distribution are likely to underestimate high percentiles, while estimates based on historical simulation face with the problem of out of sample performance. The extreme value approach seems appropriate and easy to implement. The dynamic version of the POT method, in which the conditional volatility GARCH estimation is embedded, could be useful to take into account the volatility clustering phenomenon. Finally, we should make a point of care in using extreme value theory, as we have to face some relevant issues, such as for example the scarcity of extreme data and the sensitivity of the results to the sample used.

## References

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