

Non-separability in spatiotemporal covariance models arisen by non-stationarity in time

Modelli per trattare la non separabilità spazio-temporale dovuta a non stazionarietà temporale

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Riassunto: In questo lavoro viene presentata una proposta per modellare processi spaziotemporali nell'ipotesi che l'assenza di stazionarietà temporale determini la non separabilità della funzione di covarianza spazio-temporale. In particolare, sono stati considerati aspetti legati all'assunzione di stazionarietà spaziale e all'esistenza di un processo temporale di larga scala sottostante il processo. Un modello spazio-temporale adeguato è stato costruito per dati di inquinamento da ozono troposferico.

Keywords: spatiotemporal models, non-stationarity in time, non-separability, deformation analysis, environmental applications.

1. Introduction

Spatiotemporal models are usually constructed from several components, and the inclusion of each component is performed stepwise by using the outcome of data analysis. This essay aims to develop the case of non-separability due to temporal non-stationarity. A very general spatiotemporal model is:

$$Y(t, \mathbf{x}) = \mu(t, \mathbf{x}) + Z(t, \mathbf{x}) + \epsilon(t, \mathbf{x}); t = 1, \dots, T, \mathbf{x} = 1, \dots, \mathbf{X} \quad (1)$$

where $\mu(t, \mathbf{x})$ is the spatiotemporal mean field or trend, $Z(t, \mathbf{x})$ denotes a zero mean smoothed Gaussian spatiotemporal underlying process and $\epsilon(t, \mathbf{x})$ is a Gaussian noise process, representing spatiotemporal measurement errors (and small scale variability).

The first component (the trend component) of model (1) is described in the second Section, where some methods for estimating it are outlined. Then the further formalization of a spatiotemporal process is described in the third Section, where the violation of basic assumptions such as temporal stationarity and separability are considered. In particular, spatial non-stationarity is included in the formalization of the model. In the last Section an application to environmental data is given.

2. Modelling and estimating the trend component

Studies of spatiotemporal models frequently propose the estimation of the trend component. When interest is not specially focused on trend detection, the simplest kind of

¹The research leading to this paper has been partially funded by a 2002 grant (Sector 13: Economics and Statistics, Project n.7, protocol n. 2002134337 01 for Research Project of National Interest by MIUR). The authors thank Peter Guttorp and Paul Sampson for the helpful discussion about the organization of this work.

trend component is considered: the deterministic one. Deterministic trend usually maintains the same structure in various different spatiotemporal models, whereas the underlying spatiotemporal process varies according to the phenomena in question.

The trend can be estimated by means of a two-way decomposition of the temporal and spatial effects on original data. The following expression describes a general two-effects (spatial and temporal) model for the trend:

$$\mu(t, \mathbf{x}) = \mu + \alpha(\mathbf{x}) + \beta(t), \quad (2)$$

under the constraints: $\sum_{\mathbf{x}} \alpha(\mathbf{x}) = 0$ and $\sum_t \beta(t) = 0$. In (2), μ is known as the "grand" effect, $\alpha(\mathbf{x})$ represents the spatial trend effect and $\beta(t)$ the time trend effect. To estimate these effects the least squares fit of (2) could be used. However, since means may be influenced by extreme values, medians can be used instead. That is, a fit of the same form is sought, but with $med(\alpha(\mathbf{x})) = med(\beta(t)) = 0$ and $med_{\mathbf{x}}(\epsilon(t, \mathbf{x})) = med_t(\epsilon(t, \mathbf{x})) = 0$. This is not a set of linear restrictions, so there may be various possible solutions. The Median Polish algorithm consists (see for instance Mosteller and Tukey (1977)) in augmenting the table with row and column effects, as described in detail in Bruno (2004).

3. Modelling the Z process

By studying the space-time covariance (or correlation) function, certain simplifying assumptions are usually stated, such as separability in the space-time domain and stationarity in space or/and time. These assumptions are described in detail in Bruno (2004).

This paper develops the case of non-separability due to non-stationarity on the temporal scale. This proposal is particularly suitable when consideration of an annual process may take into account features of seasonal variability besides those encoded in the spatiotemporal trend. We propose to modify the spatiotemporal model (1) as:

$$Y(t, \mathbf{x}) = \mu(t, \mathbf{x}) + \sigma_t(\mathbf{x})Z^*(t, \mathbf{x}) + \epsilon(t, \mathbf{x}); \quad t = 1, \dots, T, \quad \mathbf{x} = 1, \dots, \mathbf{X}. \quad (3)$$

In (3), parameter $\sigma_t(\mathbf{x})$ is fundamental since it represents that part of non-stationary temporal variability which, when removed, leaves separable space and time correlation components. The $\sigma_t(\mathbf{x})$ component of the model can be estimated by means of various different techniques, such as moving average windows or non parametric methods.

This re-parameterization is designed to ensure that the process $Z^*(\mathbf{x}, t)$ remains a zero-mean Gaussian process, is now a temporal stationary function, with a separable spatiotemporal covariance (correlation). In fact, we assume that non-separability derived from temporal non-stationarity in scale.

The separable spatiotemporal correlation function can be written as:

$$Corr(Z^*(t, \mathbf{x}), Z^*(s, \mathbf{y})) = \rho_1(|t - s|)\rho_2(\|\mathbf{x} - \mathbf{y}\|); \quad (4)$$

with $t, s = 1, \dots, T$, and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$.

In some cases a unique, large-scale temporal process underlying the spatiotemporal process $Z^*(t, \mathbf{x})$ in (3) is assumed. Here $Z^*(t, \mathbf{x})$ is further decomposed into two components: a (large scale) time-dependent process, denoted by $Z_1(t)$, and a temporally uncorrelated space-time process, $Z_2(t, \mathbf{x})$. Then the spatiotemporal model (3) is further developed as:

$$Y(t, \mathbf{x}) = \mu(t, \mathbf{x}) + \sigma_t(\mathbf{x})(\alpha_{\mathbf{x}}Z_1(t) + Z_2(t, \mathbf{x})) + \epsilon(t, \mathbf{x}); \quad t = 1, \dots, T, \quad \mathbf{x} = 1, \dots, \mathbf{X}. \quad (5)$$

where $\alpha_{\mathbf{x}}$ is a site-specific coefficient for the (large scale) time-dependent process $Z_1(t)$.

An additional assumption can be made when defining the spatial component of the spatiotemporal correlation function (4). In the case of many spatiotemporal processes, there is little reason to expect spatial stationarity of the covariance function. The spatial anisotropy can be analyzed by modelling non-stationary spatial covariance structures through deformations of the geographic coordinate system (Sampson and Guttorp (1992); Damian *et al.* (2001)), where the non-stationary spatial correlation function of $Z(t, \mathbf{x})$ is a function of the Euclidean distances between site locations in a bijective transformation of the geographic coordinate system:

$$\text{Corr}(Z(t, \mathbf{x}), Z(t, \mathbf{y})) = \rho_{\theta}(\|f(\mathbf{x}) - f(\mathbf{y})\|) \quad (6)$$

In (6), the symbol ρ_{θ} denotes a correlation function of a known parametric form (with unknown parameter θ) and f is an unknown function assumed to be both smooth and bijective. In this semi-parametric model, the correlation structure is isotropic (*i.e.* not dependent on direction but only on distance) after an appropriate change has been made in the coordinate system.

We propose the use of deformation analysis with regard to $Z^*(t, \mathbf{x})$, when the spatiotemporal model considered is (3), or with regard to $Z_2(t, \mathbf{x})$, when a type (5) model is assumed. In both cases, since the final spatiotemporal model has a spatiotemporal covariance function that assumes a separable structure distinguishing between temporal and spatial component, the final covariance can be formalized in a very simple way.

When Z^* (or analogously $Z_2(t, \mathbf{x})$) is introduced into the model, the spatiotemporal covariance function (4) becomes:

$$\text{Corr}(Z^*(t, \mathbf{x}), Z^*(s, \mathbf{y})) = \rho_1(|t - s|)\rho_2(\|f(\mathbf{x}) - f(\mathbf{y})\|); \quad (7)$$

with $t, s = 1, \dots, T$, and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$.

4. Application to environmental data

The data set consists of tropospheric ozone measurements from 32 monitoring stations across the Italian Emilia Romagna region. Ozone has been measured on a daily time scale as a daily maximum 8-hours moving average, computed from the hourly figures for ozone concentration recorded in micrograms per cubic meter, $\mu\text{g}/\text{m}^3$, for a 5-years period (between 1998 and 2002). Eight-hour moving average time series are suggested both by Italian Law (D.M.A., 16/05/1996) and by U.S. EPA air quality standards.

The spatiotemporal model is built along the same lines described in the previous sections. The trend component is estimated in order to include a daily effect (*i.e.* the ozone cycle over 365 days a year is very important), a yearly effect and a spatial effect. Then the spatiotemporal model (1) is initially applied to the original data. The temporal non-stationarity in scale would suggest the adoption of model (3), and estimation of the $\sigma_t(\mathbf{x})$ component by 30 days moving windows and by means of nonparametric techniques. The taking into account of the $Z_1(t)$ component in model (5) may mean that the permanence of ozone in the air for the analyzed area could be considered to be spatially uniform. This is reasonable since we know that ozone is a pollutant which behaves stably in air and which has a wide spatial range.

In order to show the advantages of choosing a flexible model, we illustrate the improvements obtained by passing from model (3), with spatiotemporal correlation function (4), to model (5), with correlation function (7).

Figure 1 shows three estimated cross-correlation functions for three respective spatiotemporal processes: $Z^*(t, \mathbf{x})$ of (3), $Z_2(t, \mathbf{x})$ of (5) and $Z_2(t, \mathbf{x})$ after the deformation analysis corresponding to (7).

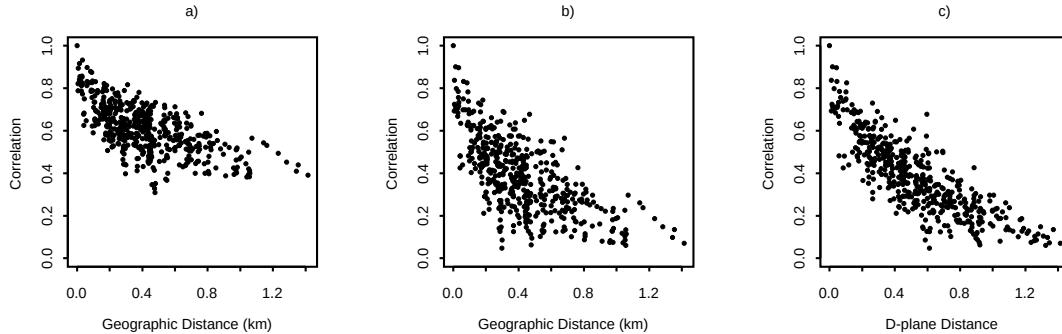


Figure 1: Display of estimated spatial correlations: left for Z^* ; center for Z_2 ; right for Z_2 after spatial deformation.

These correlations are fitted by an exponential model with a nugget effect (*i.e.* the component that denotes the non-continuity at the origin). The spatial correlations of $Z^*(t, \mathbf{x})$ are presented in the left-hand panel of Figure 1. These correlations decrease to a non-zero value for those distances represented in our data. Taking account of this feature, together with our knowledge of this pollutant's physical characteristics (its short-term stability and spatial homogeneity), would suggest the adoption of model (5) considering a large-scale process $Z_1(t)$ and a temporally uncorrelated space-time process, $Z_2(t, \mathbf{x})$. Removal of the large scale process $Z_1(t)$ enables us to obtain, in the central panel, spatial correlations for process $Z_2(t, \mathbf{x})$ tending towards zero within the spatial range of the data. Finally, the management of process anisotropy by means of the spatial deformation (Sampson and Guttorp (1992); Damian *et al.* (2001)) on $Z_2(t, \mathbf{x})$ clearly shows tighter empirical correlations (shown in the right-hand panel).

We can thus conclude that comparisons between spatiotemporal models may be based on the nugget effect of the exponential correlogram. Model (5) enables us to reduce it from 0.35 (as in the starting process $Z(t, \mathbf{x})$) to 0.22 (as in $Z_2(t, \mathbf{x})$ including anisotropy).

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