

# The Multi-State Markov Switching Model

## *Il Modello Markov-Switching Multistato*

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**Riassunto:** In molti fenomeni reali, l'evoluzione di una serie storica, soggetta a cambiamenti di regime, dipende dallo stato di un'altra serie storica, che ne anticipa il comportamento; la conoscenza di quest'ultimo può dunque aiutare a prevedere lo stato della prima serie storica. In questo lavoro viene proposto un particolare tipo di modello Markov Switching adatto a rappresentare queste situazioni; in particolare, le probabilità di transizione di una variabile di interesse dipendono non solo dal proprio stato al tempo precedente, ma anche dallo stato di altre variabili. Un modello di questo tipo equivale ad un modello Markov Switching multivariato con regimi ricavati dalla combinazione degli stati delle variabili considerate. Sono illustrati gli aspetti teorici ed una applicazione su dati relativi all'economia italiana.

**Keywords:** Regime-switching; Multivariate time series; Multiple regimes

## 1. Introduction

The Markov Switching (MS hereafter) model, introduced by Hamilton (1989), has been largely used for the study of phenomena with two or more unobservable regimes, represented by states of a Markov chain. Generally, univariate approaches are used for the applications in various fields and the extensions to multivariate cases look for common states for the same variable observed in different subjects (countries, markets, firms, etc.) or for several different variables (see, for example, Krolzig, 1997). In this paper, we deal with  $n$  variables depending on a certain number of regimes; they could be different variables or the same variable observed on  $n$  subjects. The starting point runs from the considerations that it is well difficult to suppose that the same regime structure is followed by all the  $n$  variables; this could be a convenient hypothesis, for example, for the business cycle analysis, to extract a common component representing the co-movements among variables. But in other applications, such as disease diffusion, financial markets, economic development, the regimes follow different dynamics in different variables. On the other hand, some links among the dynamics of the unobserved regimes probably exist and the regime of a variable could lead the regime of another variable.

In this paper we propose an MS model in a multivariate framework, allowing different variables representing the regimes. The dynamics of the regime of a variable depends on the dynamics of the regime of the other variables. We call this model the Multi-State Markov Switching (MSMS) model.

The aim of this paper is to formalize the MSMS model and to show its usefulness. In the next section the new model will be illustrated and in section 3 an application will

analyze the relationship existing between the turnover and new orders series with the MSMS model.

## 2. The Model Specification

Let  $y_t$  an  $n \times 1$  multivariate time series observed for  $n$  different variables. We suppose that it follows a VAR process:

$$y_t = B\mu_{S_t} + \Phi_1 y_{t-1} + \dots + \Phi_p y_{t-p} + \varepsilon_t \quad t=1, \dots, T \quad (1)$$

The vector  $\mu_{S_t}$  contains  $n$  constants and it is indicized by a multiple random variable  $S_t = (s_{1t}, s_{2t}, \dots, s_{nt})$ . We call this variable the multiple regime (or multiple state at time  $t$ ). The univariate random variable  $s_{jt}$  ( $j=1, \dots, n$ ) represents the state at time  $t$  for the variable  $j$  and can assume  $k$  possible values; for the sake of simplicity we suppose  $k=2$  and denote these states with 0 and 1 (the extension to a generic  $k$  is simple but notationally cumbersome). The  $n \times n$  matrix  $B = \{b_{ij}\}$  ( $i, j=1, \dots, n$ ) is a matrix of coefficients with elements on the diagonal equal one. In practice, the intercept of the variable  $y_{jp}$  contained in the vector  $y_p$  is equal to:

$$\mu_{s_{j,t}} + \beta_{j1}\mu_{s_{1,t}} + \dots + \beta_{jn}\mu_{s_{n,t}}$$

in which it is possible to separate an "autonomous" intercept  $\mu_{s_{j,t}}$  from the effect due to the other variables.  $\Phi_h$  ( $h=1, \dots, p$ ) is a matrix of unknown autoregressive coefficients. The  $n$ -vector of disturbances  $\varepsilon_t$  is uncorrelated along the time but correlated across the variables, to take into account their mutual influence. We suppose that  $\varepsilon_t$  is normally distributed with mean  $\mathbf{0}$  and covariance matrix with elements  $\sigma_{is_{i,t}}^2$  on the diagonal and  $\rho_{ij}\sigma_{is_{i,t}}\sigma_{js_{j,t}}$  otherwise. The number of coefficients to be estimated increases with  $n$ , so, in some cases, it will be convenient to adopt some reparameterization for the coefficients  $\rho_{ij}$ .

The multiple regime  $S_t$  is not observable and its distribution is unknown; we suppose that it follows an ergodic Markov chain. The possible realizations of  $S_t$  are  $2^n$  and the generic element of the transition probability matrix will be:

$$Pr[S_t | S_{t-1}] = Pr[s_{1t}, s_{2t}, \dots, s_{nt} | s_{1t-1}, s_{2t-1}, \dots, s_{nt-1}].$$

The states  $s_{1t}, s_{2t}, \dots, s_{nt}$ , conditional on  $S_{t-1}$ , are mutually independent, so that:

$$Pr[S_t | S_{t-1}] = Pr[s_{1t} | S_{t-1}] Pr[s_{2t} | S_{t-1}] \dots Pr[s_{nt} | S_{t-1}]. \quad (2)$$

The probabilities  $Pr[s_{jt} | S_{t-1}]$  can be parameterized with logistic functions to take into account the influence of the regimes relative to the other variables:

$$\begin{aligned}
Pr[s_{jt}=0|s_{jt-1}=0, \mathbf{S}^{(j)}_{t-1}] &= \exp(\vartheta_{j,00} + \boldsymbol{\theta}_{j,0} \mathbf{S}^{(j)}_{t-1}) / [1 + \exp(\vartheta_{j,00} + \boldsymbol{\theta}_{j,0} \mathbf{S}^{(j)}_{t-1})] \\
Pr[s_{jt}=1|s_{jt-1}=0, \mathbf{S}^{(j)}_{t-1}] &= 1 - Pr[s_{jt}=0|s_{jt-1}=0, \mathbf{S}^{(j)}_{t-1}] \\
Pr[s_{jt}=1|s_{jt-1}=1, \mathbf{S}^{(j)}_{t-1}] &= \exp(\vartheta_{j,11} + \boldsymbol{\theta}_{j,1} \mathbf{S}^{(j)}_{t-1}) / [1 + \exp(\vartheta_{j,11} + \boldsymbol{\theta}_{j,1} \mathbf{S}^{(j)}_{t-1})] \\
Pr[s_{jt}=0|s_{jt-1}=1, \mathbf{S}^{(j)}_{t-1}] &= 1 - Pr[s_{jt}=1|s_{jt-1}=1, \mathbf{S}^{(j)}_{t-1}]
\end{aligned} \tag{3}$$

In this case  $\boldsymbol{\theta}_{j,v} = (\vartheta_{j,v1}, \dots, \vartheta_{j,v\hat{g}-1}, \vartheta_{j,v\hat{g}+1}, \dots, \vartheta_{j,vn})'$  ( $v=0,1$ ) is a  $(n-1)$ -vector of unknown parameters and  $\mathbf{S}^{(j)}_{t-1}$  is equal to the multiple regime vector  $\mathbf{S}_{t-1}$ , excluding the  $j$ -th element  $s_{jt-1}$ . Using the expressions in (3) it is possible to estimate the conditional probabilities for each regime and the full matrix with elements expressed by (2). To estimate the model (1)-(2)-(3) we use the procedure of filtering proposed by Kim (1994). Note that the estimates of the probabilities  $Pr[s_{jt}|\mathbf{S}_{t-1}]$  and the transition probabilities matrix  $\mathbf{P} = \{Pr[\mathbf{S}_t|\mathbf{S}_{t-1}]\}$  provide additional information: the former show how the probability to stay in the same regime or to change it for the variable  $j$  changes when the regimes relative to the other variables change; the latter indicate what is the most probable scenario (a particular combination of regimes for the various variables) at time  $t$  given a certain scenario at time  $t-1$ .

### 3. An Example: an Application to the Italian Turnover Index

It is a common practice to consider the industrial new orders as a leading indicator of the industrial turnover. We consider the monthly trend-cycle series of indices of the Italian new orders and turnover from January 1990 to June 2002 (elaboration on Istat data). To take into account this characteristic, we specify an MSMS model in which the new orders can determine the level and the regime of the turnover, but not vice-versa. Labeling with 1 the turnover and 2 the new orders, we use a model like (1), with  $p=1$  and  $\boldsymbol{\Phi}_1 = \text{diag}(\phi_{1,1}, \phi_{2,1})$ , in which the only element of the matrix  $\mathbf{B}$  different by 1 is the element  $\beta_{12}$ . For the estimation of the transition probabilities matrix we use the parameterization (3); using our priors on the dependence relationships, they are parameterized as:

$$\begin{aligned}
Pr[s_{1t}=0|s_{1t-1}=0, s_{2t-1}] &= \exp(\vartheta_{1,00} + \vartheta_{1,02}s_{2t-1}) / [1 + \exp(\vartheta_{1,00} + \vartheta_{1,02}s_{2t-1})] \\
Pr[s_{2t}=0|s_{2t-1}=0, s_{1t-1}] &= \exp(\vartheta_{2,00}) / [1 + \exp(\vartheta_{2,00})] \\
Pr[s_{1t}=1|s_{1t-1}=1, s_{2t-1}] &= \exp(\vartheta_{1,10} + \vartheta_{1,12}s_{2t-1}) / [1 + \exp(\vartheta_{1,10} + \vartheta_{1,12}s_{2t-1})] \\
Pr[s_{2t}=1|s_{2t-1}=1, s_{1t-1}] &= \exp(\vartheta_{2,10}) / [1 + \exp(\vartheta_{2,10})]
\end{aligned} \tag{4}$$

The maximum likelihood (ML) estimates of the coefficients are shown in Table 1 (the asymptotic theory for ML estimators of MS models is by no means standard, but, under given conditions, they are consistent and asymptotically Normal, Francq and Roussignol, 1997). Note that the intercept of the turnover index ( $\mu_{1h} + \beta_{12}\mu_{2r}$ ,  $h, r=0, 1$ ) changes with the regimes  $s_{1t}$  and  $s_{2t}$ , adapting to the various scenarios; in particular it will be -0.33, -0.01, 0.24, 0.56. In Table 2 are shown the probabilities expressed in (4) (we indicate with  $p^{(i)}_{i,hr}$  the probability  $Pr[s_{jt}=i|s_{1t-1}=h, s_{2t-1}=r]$ ). The influence of  $s_{2t}$  on the changes in regime for turnover are evident; in fact, the probability of variable 1 to

stay in regime 0, when the variable 2 was in regime 0 in the previous month, is 0.95, but decreases to 0.48 when variable 2 was in regime 1. In a similar way, the turnover remains in regime 1 with a 86% of chances, when the new orders were in the same regime, but switches to regime 0 with a probability equal to 47% when the variable 2 was in regime 0. The last two columns of Table 2 show the transition probabilities for the variable 2, that do not depend on  $s_{1t-1}$ , as shown in (4).

In Table 3 the transition probability matrix is shown ( $p_{ih,rv}$  indicates the probability  $Pr[s_{1t}=i, s_{2t}=h|s_{1t-1}=r, s_{2t-1}=v]$ ). When both the variables are in the same state at time  $t-1$ , it is probable to maintain this scenario in the next month. When the new orders index is in a different state with respect to the turnover at time  $t-1$ , the most probable result at time  $t$  will be the persistence of this scenario or the alignment of turnover to the same regime of new orders.

**Table 1:** Estimates of the MSMS model and relative standard errors

| $\mu_{10}$    | $\mu_{11}$    | $\mu_{20}$ | $\mu_{21}$         | $\beta_{12}$       | $\phi_{1,1}$       | $\phi_{2,1}$       | $\sigma_{10}$      | $\sigma_{11}$      |
|---------------|---------------|------------|--------------------|--------------------|--------------------|--------------------|--------------------|--------------------|
| -0.06         | 0.51          | -0.04      | 0.23               | -1.16              | 0.75               | 0.77               | 0.14               | 0.11               |
| (0.05)        | (0.05)        | (0.04)     | (0.06)             | (0.40)             | (0.02)             | (0.03)             | (0.02)             | (0.01)             |
| $\sigma_{20}$ | $\sigma_{21}$ | $\rho$     | $\vartheta_{1,00}$ | $\vartheta_{1,02}$ | $\vartheta_{2,00}$ | $\vartheta_{1,10}$ | $\vartheta_{1,12}$ | $\vartheta_{2,10}$ |
| 0.31          | 0.43          | 0.66       | 2.89               | -2.95              | 1.45               | 0.12               | 1.71               | 1.16               |
| (0.02)        | (0.03)        | (0.04)     | (0.84)             | (1.11)             | (0.49)             | (0.63)             | (1.02)             | (0.35)             |

**Table 2:** Conditional probabilities for the regimes of each series

| $p^{(1)}_{0,00}$ | $p^{(1)}_{0,01}$ | $p^{(1)}_{1,10}$ | $p^{(1)}_{1,11}$ | $p^{(2)}_{0,00}=p^{(2)}_{0,10}$ | $p^{(2)}_{1,01}=p^{(2)}_{1,11}$ |
|------------------|------------------|------------------|------------------|---------------------------------|---------------------------------|
| 0.947            | 0.485            | 0.531            | 0.862            | 0.809                           | 0.762                           |

**Table 3:** Estimated transition probabilities matrix

|              | $p_{00,..}$ | $p_{01,..}$ | $p_{10,..}$ | $p_{11,..}$ |
|--------------|-------------|-------------|-------------|-------------|
| $p_{,..,00}$ | 0.767       | 0.181       | 0.042       | 0.010       |
| $p_{,..,01}$ | 0.115       | 0.369       | 0.123       | 0.393       |
| $p_{,..,10}$ | 0.380       | 0.089       | 0.430       | 0.101       |
| $p_{,..,11}$ | 0.033       | 0.105       | 0.205       | 0.657       |

## References

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