

Aggregative Index Numbers Minimal System

Sistema Minimale di Numeri Indice Aggregativi

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Riassunto: In questo ultimo secolo, gli aggregati su cui si calcolano gli indici sono cambiati sia in termini di beni che di rilevanza economica (e.g. mercati finanziari), contemporaneamente, le nuove organizzazioni economiche regionali e sovranazionali (e.g. Unione Europea) hanno posto nuove domande sia in termini di classificazione che in termini di misure non solo settoriali delle attività economiche. L'evoluzione tecnologica, infine, ha comportato sia la disponibilità immediata delle informazioni che la presenza sistematica di "missing values" nelle basi di dati. Il sistema minimale di numeri indice aggregativi, proposto, risponde ai requirements imposti dalla modernità e propone contestualmente una nuova notazione, che grazie all'impiego della funzione indicatrice, consente di ottenere misure corrette degli aggregati trattati.

Keywords: Aggregative Index Numbers Minimal System (F-mSystem), aggregative indices, value index, Laspeyres index, Paasche index, Fisher index, representativeness.

1. Introduction and aims

In the theory of the numbers index, the methodological approaches, matured in the course of the twentieth century, are subdivided into two fundamental groups: the Fisher approach (see Fisher, 1922) and the axiomatic approach (see Eichhorn, Voeller 1976; Vogt 1979; Martini 1992, 2001).

The axiomatic approach also places itself in the Fisherian tradition, but it turns over the formulation: from the *ex post* verification of the properties it arrives, like in other disciplines, at the *ex ante* definition of a function that must satisfy specific axioms that formalize the fundamental principles of the theory. The common aspects of the two approaches, as it is well known, are the focus on the "*price index*" and the research of the "*best*" formula. The aim of this work is to introduce the Aggregative Index Number Minimal System (F-mSystem), within the axiomatic formulation, but based on the Value Index and on its "*aggregative*" factorizations. This new approach determines a more restrictive set of indices compared with the Martinian sets.

2. A new notation

The tradition in the index numbers context, from its origin, has been characterized by the focus on the price index and by the research of the "*best*" formula. This focus often leads to miscalculations of the index. In contrast with the tradition, a new layout is suggested, which focuses on the value index and proposes, as its factorizations, indices able to explain variations contributing to prices and to quantities. In this way, the

correct value index used is common to both situations. If the base period is relatively close, there are no representativeness problems; in fact the fixed-base total value index and the one common to both situations will not be very different.

The F-mSystem is a well-defined indices system, i.e. where the value index can be factorized on contribution due to prices and to quantities. It is an indices system able to provide a full information on a measured aggregate (e.g. indices of which we know the representativeness). It is an aggregative indices system, i.e. from a set of sub-sets to indices of sub-indices. The F-mSystem is based on:

- new focus: the Value Index (not the Price Index);
- new essential axiomatic properties: Aggregativity (as regards Martini, 2001);
- new approach: System of Indices (not the “best” formula approach);
- new formulas set: real factorization of value index on contribution due to prices and to quantities variation;
- indicator function for building the indices common to both situations;
- fixed-base formula.

As well known if an index and its cofactor do not satisfy one of the essential axiomatic properties they cannot be considered proper measures of prices and quantities variations. The irrenounceable properties of a number index, in an axiomatic approach are the Commensurability (Co), the Monotonicity (Mo), the Homogeneity (Ho) and the Associativity (As) (see Martini, 1992). The F-mSystem is in the Martini tradition, but it focuses on the value index, and requires that its indices satisfy one more essential property: the aggregativity (or indirect aggregativity). The aggregativity (Ag) property implies the associativity (see Martini, 2001).

Definition: Indirect Aggregativity (AgI) in the systemic approach

A Crossing index of aggregates indices, in the systemic approach, is itself aggregative (indirectly).

Definition: Number Index in the systemic approach

Let $(\Omega, \mathcal{F})$ be a measurable space, $\Omega \subseteq \mathcal{R}^+$ a set called the *sample space*, $\mathcal{F}$ a σ -algebra on Ω , ${}^c I_{[0 \cap t]}$ the indicator function common to both situations (at time 0 and t).

A non-negative map defined on $\mathcal{F}$

$$f \cdot {}^c I_{[0 \cap t]}: \mathcal{F} \rightarrow [0, \infty) \quad (1)$$

is called a Number Index on $(\Omega, \mathcal{F})$ if it satisfies the following properties:

- i) $f(p_0, p_t, q_0, q_t) \cdot {}^c I_{[0 \cap t]} \in \text{Co}$
- ii) $f(p_0, p_t, q_0, q_t) \cdot {}^c I_{[0 \cap t]} \in \text{Ho}$
- iii) $f(p_0, p_t, q_0, q_t) \cdot {}^c I_{[0 \cap t]} \in \text{Mo}$
- iv) $f(p_0, p_t, q_0, q_t) \cdot {}^c I_{[0 \cap t]} \in \text{Ag (or AgI)}$

and if its cofactor (*cof*) is a non-negative map defined on $\mathcal{F}$

$$\text{Cof}(f \cdot {}^c I_{[0 \cap t]}): \mathcal{F} \rightarrow [0, \infty) \quad (2)$$

that satisfies the following properties

- v) $\text{Cof}(f(p_0, p_t, q_0, q_t) \cdot {}^c I_{[0 \cap t]}) \in \text{Ho}$
- vi) $\text{Cof}(f(p_0, p_t, q_0, q_t) \cdot {}^c I_{[0 \cap t]}) \in \text{Mo}$
- vii) $\text{Cof}(f(p_0, p_t, q_0, q_t) \cdot {}^c I_{[0 \cap t]}) \in \text{Ag (or AgI)}.$

In the systemic approach, also the Ag (or AgI) becomes an essential property¹ because it is necessary for the aggregation of indices of the different sets.

3. Building the Aggregative Index Numbers Minimal System

A discrete-time model is built on finite probability space $(\Omega, \mathcal{F}, P)$ where $\mathcal{F}$ is a σ -algebra on Ω (collection of sample points) and P is a probability measure on $(\Omega, \mathcal{F})$. Let $\{\mathcal{F}_t\}_{t \in \mathbb{N}}$ be a filtration (a non decreasing family of sub σ -algebras of $\mathcal{F}$) where $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \dots \subseteq \mathcal{F}$ for $\forall t \in \mathbb{N}$, it models the information available to us at time t . For example, the market consists in $s+1$ securities, whose values at time t are given by the non-negative variables $v_{t,0}, v_{t,1}, \dots, v_{t,s}$, measurable with respect to $\mathcal{F}_t$ (we know past and present values but not the future ones). Let $v_{0,0,z} = p_{0,z} q_{0,z}$ be the z^{th} securities value and let $V_{00} = \sum_{z \in \mathbb{N}} v_{0,0,z}$ be the Stock exchange value at time 0. Let $v_{t,0,z} = p_{t,z} q_{t,z}$ be the z^{th} securities value and let $V_{tt} = \sum_{z \in \mathbb{N}} v_{t,0,z}$ be the Stock exchange value at time t . Let $I_{z[0]}$ and $I_{z[t]}$ be the z^{th} securities indicator functions² respectively at time 0 and t . Let ${}^c I_{z[0 \cap t]}$ be the z^{th} security indicator function³ common to both situations (at time 0 and t). Let ${}^c V_{00} = \sum_{z \in \mathbb{N}} [v_{0,0,z} \cdot {}^c I_{z[0 \cap t]}]$ be the Stock exchange value common to both situations at time 0 and let ${}^c V_{tt} = \sum_{z \in \mathbb{N}} [v_{t,0,z} \cdot {}^c I_{z[0 \cap t]}]$ be the Stock exchange value common to both situations at time t . Let ${}_0^c V_t = {}^c V_{tt} / {}^c V_{00}$ be the value index common to both situations. Let ${}_0 V_t = V_{tt} / V_{00}$ be the total value index, but, as time goes by, a group of shares come out of the basket which determines the unavailability of the entire series in the next periods so the calculable total value index becomes ${}_0 V_t = V_{tt} / {}^c V_{00}$.

Let ${}^c v_{0,t,z} = [p_{0,z} q_{t,z}] \cdot {}^c I_{z[0 \cap t]}$ be the z^{th} securities value at time 0 with the t quantity and ${}^c V_{0t}$ its sum total; let ${}^c v_{t,0,z} = [p_{t,z} q_{0,z}] \cdot {}^c I_{z[0 \cap t]}$ be the z^{th} securities value at time t with the 0 quantity and ${}^c V_{t0}$ its sum total. We have the following expressions:

$$\begin{aligned} {}_0^c P_t^L &= {}^c V_{t0} / {}^c V_{00} && \text{Laspeyres price index number common to both situations;} \\ {}_0^c P_t^P &= {}^c V_{tt} / {}^c V_{0t} && \text{Paasche price index number common to both situations;} \\ {}_0^c Q_t^L &= {}^c V_{0t} / {}^c V_{00} && \text{Laspeyres quantity index number common to both situations;} \\ {}_0^c Q_t^P &= {}^c V_{tt} / {}^c V_{t0} && \text{Paasche quantity index number common to both situations;} \\ {}_0^c P_t^F &= ({}_0^c P_t^L \cdot {}_0^c P_t^P)^{1/2} && \text{Fisher price index number common to both situations;} \\ {}_0^c Q_t^F &= ({}_0^c Q_t^L \cdot {}_0^c Q_t^P)^{1/2} && \text{Fisher quantity index number common to both situations.} \end{aligned}$$

The value index (${}_0^c V_t$) can be factorized on contribution due to prices and to quantities:

$${}_0^c V_t = {}^c V_{tt} / {}^c V_{00} = {}_0^c P_t^L \cdot {}_0^c Q_t^P = {}_0^c P_t^P \cdot {}_0^c Q_t^L = {}_0^c P_t^F \cdot {}_0^c Q_t^F \quad (3)$$

¹ From the systemic definition of index numbers other properties follows. For the Index:

- i) $f \in \text{Ag} \Rightarrow f \in \text{As} \Rightarrow f \in \text{PrF}$
- ii) $f \in \text{AgI} \Rightarrow f \in \text{As} \Rightarrow f \in \text{PrF}$ (PrF: Proportionality)

and for the cofactor of the index:

- iii) $\text{cof}(f) \in \text{Ag} \Rightarrow \text{cof}(f) \in \text{As} \Rightarrow \text{cof}(f) \in \text{PrF}$
- iv) $\text{cof}(f) \in \text{AgI} \Rightarrow \text{cof}(f) \in \text{As} \Rightarrow \text{cof}(f) \in \text{PrF}$

² $I_{z[0]} = \begin{cases} 1 & \text{for } v_{0,0,z} > 0 \\ 0 & \text{for } v_{0,0,z} = 0 \end{cases} \quad I_{z[t]} = \begin{cases} 1 & \text{for } v_{t,0,z} > 0 \\ 0 & \text{for } v_{t,0,z} = 0 \end{cases}$

³ ${}^c I_{z[0 \cap t]} = I_{z[0]} \cdot I_{z[t]}$

4. Conclusion

The Aggregative Index Numbers Minimal System is based on the Value Index and the Fisher, Laspeyres, and Paasche true formulas i.e. the indices are built by using the Indicator Function. In order to account for the modern data structure (missing values) therefore, we need to modify the usual number index formulas. For instance the Laspeyres price index can be adjusted as

$${}^{MR}_0 P_{t, \frac{p_{0,z} q_{0,z}}{\sum_z p_{0,z} q_{0,z}}} = \sum_z \left[\frac{p_{t,z}}{p_{0,z}} \left(\frac{p_{0,z} q_{0,z}}{\sum_z p_{0,z} q_{0,z}} \bullet {}^c I_{z[0 \leadsto t]} \right) \right] \quad (4)$$

In the systemic approach only few number indices (with their cofactor), that satisfy the essential axiomatic properties, are also aggregatives. In particular:

- Laspeyres and Paasche are Co, Ho, Mo, Ag (obviously also each cofactor has the same properties);
- *cof*(Fisher) and Fisher (1922) are Co, Ho, Mo, AgI (see Verrecchia, 2003), AnB, AnF;
- *cof*(Martini) and Martini (1992) are Co, Ho, Mo, Ag, AnB, AnF.

Instead different famous indices are not aggregative, for example:

- *cof*(Sato-Vartia) and Sato-Vartia (1974) are Co, Ho, Mo, As, AnB, AnF;
- Walsh (1901) is Co, Ho, Mo, Ag, AnB but *cof*(Walsh) is Co, Ho, Mo, AnB;
- Diewert⁴ (1976) is Co, Ho, Mo, AnB and *cof*(Diewert) is Co, Ho, AnB.

References

- Fisher I. (1967) *The making of index numbers: a study of their Varieties, Tests, and Reliability*, reprinted by Augustus M. Kelley Publishers, New York.
- Koves P. (1983) *Index theory and economic reality*, Akademiai Kiado, Budapest.
- Lamberton D., Lapeyre B. (2000) *Introduction to Stochastic calculus Applied to Finance*, Chapman & Hall/CRC, New York.
- Martini M. (1992) *I numeri indice in un approccio assiomatico*, Giuffrè, Milano.
- Martini M. (2001) *Numeri indice per il confronto nel tempo e nello spazio*, CUSL, Milano, 213-230.
- Pliska S. R. (1997) *Introduction to mathematical Finance: discrete time models*, Blackwell Publishers Ltd, Oxford.
- Verrecchia F., Zavanella B. (2002) Methodological Problems in Index Numbers' Construction For Multi-Temporal Comparison in the Financial Field: *Atti della XLI Riunione Scientifica convegno SIS2002*, Milano, 529-532.
- Verrecchia F. (2003), Index numbers system for spatial and time comparison applied to finance: *Atti del Convegno Intermedio SIS*, Napoli.
- Williams D. (1991) *Probability with Martingales*, Cambridge University press, Cambridge.

⁴ The Diewert index satisfies the essential axiomatic properties with $\rho=2$.