

Threshold Bipower Variation and the Impact of Jumps on Volatility Forecasting

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Abstract: This study reconsiders the role of jumps for volatility forecasting by showing that jumps have a positive and mostly significant impact on future volatility. This result becomes apparent once volatility is correctly separated into its continuous and discontinuous component. To this purpose, we introduce the concept of threshold bipower variation, which is based on the joint use of bipower variation and threshold estimation. We show that its generalization (threshold multipower variation) admits a feasible central limit theorem in the presence of jumps, which is not attainable for standard multipower variation if not in very restrictive cases. Importantly, with respect to the standard multipower variation, our estimator provides less biased estimates of the continuous quadratic variation and jumps in finite samples. It also provides a new test for jump detection which has substantially more power than previous tests. Empirical analysis (on the S&P500 index, single stocks and US bond yields) shows that the proposed techniques improve significantly the accuracy of volatility forecasts especially in periods following the occurrence of a jump.

Keywords: volatility estimation, jump detection, volatility forecasting, threshold estimation, financial markets

1. Contributions of the paper

In this study we start from an apparent puzzle contained in the study of Andersen *et al.* (2007) and Forsberg and Ghysels (2007); Giot and Laurent (2007); Busch *et al.* (2006). In these works, jumps have been found to possess a negative or null impact in determining future volatility, which appears puzzling.

This paper provides four contributions to the literature. The first contribution is to show that this puzzle is due to the use of the bipower variation (Barndorff-Nielsen and Shephard, 2004) to estimate the continuous integrated volatility and, by difference, the jump contribution to total quadratic variation. While bipower variation is a consistent estimator of integrated volatility, our simulation studies show that in finite samples it is largely upper biased in presence of jumps, and this implies a large underestimation of the jump component. Unfortunately, this problem cannot be accommodated by simply shrinking the

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observation interval, since market microstructure effects would jeopardize the estimation in an unpredictable way.

The second contribution of this paper is to introduce an alternative estimator of integrated powers of volatility in presence of jumps. We introduce the concept of *threshold multipower variation*, which can be viewed as combination of multipower variation (Barndorff-Nielsen and Shephard, 2006) and the threshold method of Mancini (2009). We provide for it a full asymptotic theory and, importantly, we obtain a central limit theorem in presence of jumps, which is not possible for multipower variation if not in very restrictive cases. Using realistic simulation of asset prices, we show that threshold bipower variation is nearly unbiased on continuous trajectories and, importantly, also in the presence of jumps. Moreover, it is robust to the choice of the threshold function, in the sense that the impact of the threshold on estimation is marginal. Thus, it is an ideal candidate to estimate dynamic models of volatility in which we use separately the continuous and discontinuous volatility as explanatory variables.

Our third contribution is the introduction of a novel test for jump detection in time series. Our test is basically a correction of the z statistics of Barndorff-Nielsen and Shephard (2006) based on threshold multipower variation. This correction removes the finite-sample bias in estimating the integral of the second and fourth power of continuous volatility in presence of jumps. We show that the proposed C-Tz test is sized as the traditional z -test under the null. In presence of jumps instead, the corrected test has significantly more power than the z test, especially when jumps are consecutive, a situation which is quite frequent in high-frequency data. We show this on simulated data.

Empirical results, based on the HAR model of Corsi (2009), confirm our theoretical results and constitute our fourth contribution. We work on stock index futures, single stocks and treasury bond data. On equity data, we find that jumps have a positive and highly significant impact on future volatility, a result which cannot be observed when bipower variation is employed because of its inherent bias. Uniformly on our data sets, our forecasting models provide higher R^2 , especially in days following a jump, and generally significant lower relative root mean square error, with respect to the existing models in the literature.

2. Main results

Assumption $(X_t)_{t \in [0, T]}$ is a real-valued process such that X_0 is measurable with respect to $\mathcal{F}_0$ and

$$dX_t = \mu_t dt + \sigma_t dW_t + dJ_t \quad (1)$$

where μ_t is predictable, σ_t is càdlàg; $dJ_t = c_t dN_t$ where N_t is a non-explosive Poisson process whose intensity is an adapted stochastic process λ_t , the times of the corresponding jumps are $(\tau_j)_{j=1, \dots, N_T}$ and c_j are i.i.d. adapted random variables measuring the size of the jump at time τ_j and satisfying, $\forall t \in [0, T]$, $\mathcal{P}(\{c_j = 0\}) = 0$.

We now introduce the following extension of multipower variation. In what follows, we use a strictly positive random threshold function $\vartheta_s : [t, t + T] \rightarrow \mathbb{R}^+$, see e.g. Mancini and Renò (2008). We define the threshold multipower variation estimator as follows:

Definition Let $\gamma_1, \dots, \gamma_M > 0$. We define the realized threshold multipower variation as:

$$\text{TMPV}_\delta(X)_t^{[\gamma_1, \dots, \gamma_M]} = \delta^{1 - \frac{1}{2}(\gamma_1 + \dots + \gamma_M)} \sum_{j=M}^{[T/\delta]} \prod_{k=1}^M |\Delta_{j-k+1} X|^{\gamma_k} I_{\{|\Delta_{j-k+1} X|^2 \leq \vartheta_{j-k+1}\}} \quad (2)$$

A simple special case of TMPV is *threshold bipower variation*, obtained with $\gamma_1 = \gamma_2 = 1$:

$$\text{TBPV}_\delta(X)_t = \mu_1^{-2} \text{TMPV}_\delta(X)_t^{[1,1]} = \mu_1^{-2} \sum_{j=2}^{[T/\delta]} |\Delta_{j-1}X| \cdot |\Delta_jX| I_{\{|\Delta_{j-1}X|^2 \leq \vartheta_{j-1}\}} I_{\{|\Delta_jX|^2 \leq \vartheta_j\}} \quad (3)$$

Importantly, threshold multipower variation admits a central limit theorem even in presence of jumps, while multipower variation admits a central limit theorem only with tight restrictions on the $\gamma_1, \dots, \gamma_M$ (which exclude bipower variation, for example). This is stated in the following theorem:

Theorem 1 *Assume that $\vartheta_t = \xi_t \Theta(\delta)$, where $\Theta(\delta)$ is a real function vanishing slower than the modulus of continuity of the Brownian motion, and ξ_t is a stochastic process on $[0, T]$ which is a.s. bounded and with a strictly positive lower bound. Then, as $\delta \rightarrow 0$:*

$$\text{TMPV}_\delta(X)_t^{[\gamma_1, \dots, \gamma_M]} \xrightarrow{p} \left(\prod_{k=1}^M \mu_{\gamma_k} \right) \int_t^{t+T} \sigma_s^{\gamma_1 + \dots + \gamma_M} ds \quad (4)$$

where the above convergence is in probability, and

$$\delta^{-\frac{1}{2}} \left(\text{TMPV}_\delta(X)_t - \int_t^{t+T} \sigma_s^{\gamma_1 + \dots + \gamma_M} ds \right) \longrightarrow c_\gamma \int_t^{t+T} \sigma_s^{\gamma_1 + \dots + \gamma_M} dW'_s \quad (5)$$

where W' is a Brownian motion independent on W , the above convergence holds stably in law and c_γ^2 is a constant.

We can exploit the estimate of the continuous local volatility given by the threshold itself replacing σ^2 by ϑ/c_ϑ^2 . Then, we define the corrected realized threshold multipower estimator as:

$$\text{C-TMPV}_\delta(X)_t^{[\gamma_1, \dots, \gamma_M]} = \delta^{1-\frac{1}{2}(\gamma_1 + \dots + \gamma_M)} \sum_{j=M}^{[T/\delta]} \prod_{k=1}^M Z_{\gamma_k}(\Delta_{j-k+1}X, \vartheta_{j-k+1}) \quad (6)$$

where the function $Z_\gamma(x, y)$ is defined as:

$$Z_\gamma(x, y) = \begin{cases} |x|^\gamma & \text{if } x^2 \leq y \\ \frac{1}{2N(-c_\vartheta)\sqrt{\pi}} \left(\frac{2}{c_\vartheta^2} y \right)^{\frac{\gamma}{2}} \Gamma\left(\frac{\gamma+1}{2}, \frac{c_\vartheta^2}{2}\right) & \text{if } x^2 > y \end{cases} \quad (7)$$

For example, the corrected version of (3) is the corrected threshold bipower variation defined as:

$$\text{C-TBPV}_\delta(X)_t = \mu_1^{-2} \text{C-TMPV}_\delta(X)_t^{[1,1]} = \mu_1^{-2} \sum_{j=2}^{[T/\delta]} Z_1(\Delta X_j, \vartheta_j) Z_1(\Delta X_{j-1}, \vartheta_{j-1}) \quad (8)$$

As will be shown in the next section, the correction is essential for building test statistics; indeed, the correction provides unbiased estimates under the null. The test statistics which is used in our empirical analysis is based on this correction and it is defined by:

$$\text{C-Tz} = \delta^{-\frac{1}{2}} \frac{(\text{RV}_\delta(X)_T - \text{C-TBPV}_\delta(X)_T) \cdot \text{RV}_\delta(X)_T^{-1}}{\sqrt{\left(\frac{\pi^2}{4} + \pi - 5\right) \max \left\{ 1, \frac{\text{C-TTriPV}_\delta(X)_T}{(\text{C-TBPV}_\delta(X)_T)^2} \right\}}}, \quad (9)$$

It is immediate to prove the following:

Corollary 2 *Under the assumptions of Theorem 1 we have that if $dJ_t = 0$ then $C-Tz \rightarrow \mathcal{N}(0, 1)$ stably in law as $\delta \rightarrow 0$.*

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