

The analysis of electric power price data using maximum likelihood and a multiscale stochastic volatility model^{†*}

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Abstract: In this paper a formulation of the calibration problem for a multiscale stochastic volatility model based on filtering and maximum likelihood methods is presented. Real data (electric power price data) are used to validate the formulation of the calibration problem and its numerical solution.

Keywords: multiscale stochastic volatility models, filtering theory, calibration problem.

We use filtering and maximum likelihood methods to solve a calibration problem for a multiscale stochastic volatility model. This work extends the work presented in [1]. The multiscale stochastic volatility model considered generalizes the Heston model, has been introduced in [2], and describes the dynamics of the asset price using as auxiliary variables two stochastic variances varying on two different time scales. Our aim is to estimate the parameters of this multiscale model and its two initial stochastic variances from the knowledge, at discrete times of the asset price and eventually of the prices of European options. This problem is translated in a maximum likelihood problem with the likelihood function defined through the solution of a filtering problem. Furthermore we develop a tracking procedure that is able to track the asset price and the values of its two stochastic variances for time values where there are no data available. The solution of the calibration problem and the tracking procedure are used to do the analysis of data time series taken from the U.S. electricity market. In the study of electric power price data we consider daily data. Moreover we use the estimated values of the parameters and of the initial stochastic variances obtained solving the calibration problem to produce, through the tracking procedure mentioned above, forecasts of the asset prices (electric power) and of the associated stochastic variances. The forecasts of the asset prices are compared with the prices actually observed. This comparison shows that the forecasts are of very high quality even when we consider “spiky” electric power price data (see Figure 1). The website: <http://www.econ.univpm.it/recchioni/finance/w9> contains some auxiliary material that helps the understanding of this paper and of [1].

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The model considered describes the dynamics of the log-return $x_t, t \geq 0$, of an asset price $S_t, t \geq 0$ (i.e.: $x_t = \ln(S_t/S_0)$), and of its two stochastic variances $v_{i,t}, t \geq 0, i = 1, 2$, and consists in the following system of Ito stochastic differential equations:

$$dx_t = (\hat{\mu} + a_1 v_{1,t} + a_2 v_{2,t}) dt + b_1 \sqrt{v_{1,t}} dW_t^{0,1} + b_2 \sqrt{v_{2,t}} dW_t^{0,2}, \quad t > 0, \quad (1)$$

$$dv_{1,t} = \chi_1(\theta_1 - v_{1,t}) dt + \varepsilon_1 \sqrt{v_{1,t}} dW_t^1, \quad t > 0, \quad (2)$$

$$dv_{2,t} = \chi_2(\theta_2 - v_{2,t}) dt + \varepsilon_2 \sqrt{v_{2,t}} dW_t^2, \quad t > 0, \quad (3)$$

with initial conditions:

$$x_0 = \tilde{x}_0, \quad v_{1,0} = \tilde{v}_{1,0}, \quad v_{2,0} = \tilde{v}_{2,0}, \quad (4)$$

where $W_t^{0,1}, W_t^{0,2}, W_t^1, W_t^2, t > 0$, are standard Wiener processes such that $W_t^{0,1} = W_t^{0,2} = W_0^1 = W_0^2 = 0$, and $dW_t^{0,1}, dW_t^{0,2}, dW_t^1, dW_t^2, t > 0$, are their stochastic differentials. The autocorrelation coefficients of these stochastic differentials are equal to one, moreover we assume that $\langle dW_t^1 dW_t^2 \rangle = 0, \langle dW_t^{0,1} dW_t^1 \rangle = \rho_{0,1} dt, \langle dW_t^1 dW_t^2 \rangle = 0, \langle dW_t^{0,1} dW_t^1 \rangle = \rho_{0,1} dt, \langle dW_t^{0,1} dW_t^2 \rangle = 0, \langle dW_t^{0,2} dW_t^1 \rangle = 0, \langle dW_t^{0,2} dW_t^1 \rangle = 0, \langle dW_t^{0,2} dW_t^2 \rangle = \rho_{0,2} dt, \langle dW_t^{0,1} dW_t^{0,2} \rangle = 0, t > 0$, where $\langle \cdot \rangle$ denotes the expected value of $\cdot$, $\rho_{0,1}, \rho_{0,2} \in [-1, 1]$ are constants known as correlation coefficients and the quantities $a_i, b_i, \chi_i, \varepsilon_i, \theta_i, i = 1, 2$, are real constants. We require $\chi_i \geq 0, \varepsilon_i \geq 0, \theta_i \geq 0, i = 1, 2$, due to elementary considerations and $\frac{2\chi_i \theta_i}{\varepsilon_i^2} > 1$ in order to

guarantee that when $v_{i,t}$ solution of (2), or of (3), is positive with probability one at time $t = 0$, it remains positive with probability one for $t > 0, i = 1, 2$. As suggested by the analogy with the Heston model we choose $a_1 = a_2 = -1/2, b_1 = b_2 = 1$. The two stochastic variances $v_{i,t}, i = 1, 2, t \geq 0$, vary on two different time scales and can be regarded as auxiliary variables introduced to model the dynamics of the asset price. The fact that $v_{1,t}, v_{2,t}, t > 0$, are stochastic variances on different time scales is translated in the condition: $\chi_1 \ll \chi_2$. In [2] the transition probability density function p_f of the stochastic process solution of the model (1), (2), (3) has been written as a one dimensional integral of an explicitly known integrand. Moreover we have defined the risk neutral measure associated to the multiscale stochastic volatility model (1), (2), (3) and the associated risk premium parameters $\lambda_1, \lambda_2 \in \mathbb{R}$ and we have derived two integral formulae that give the price of European vanilla call and put options under the risk neutral measure [2]. Let $\mathbb{R}$ be denote the set of real numbers. In order consider the filtering and calibration problems that interest us we introduce the vector $\underline{\Theta} \in \mathbb{R}^{13}$ and the set $\mathcal{M} \subset \mathbb{R}^{13}$ defined as follows:

$$\mathcal{M} = \{ \underline{\Theta} = (\varepsilon_1, \theta_1, \rho_{0,1}, \chi_1, \tilde{v}_{0,1}, \hat{\mu}, \lambda_1, \varepsilon_2, \theta_2, \rho_{0,2}, \chi_2, \tilde{v}_{0,2}, \theta, \lambda_2) \in \mathbb{R}^{13} \mid$$

$$\varepsilon_i, \chi_i, \theta_i, \geq 0, \frac{2\chi_i\theta_i}{\varepsilon_i^2} \geq 1, -1 \leq \rho_{0,i} \leq 1, \tilde{v}_{0,i} \geq 0, \chi_i + \lambda_i > 0, i = 1, 2 \left. \vphantom{\frac{2\chi_i\theta_i}{\varepsilon_i^2}} \right\}. \quad (5)$$

The vector $\underline{\Theta} \in \mathbb{R}^{13}$ is the unknown of the calibration problem considered and the set $\mathcal{M}$ defines the set of “feasible” vectors, that is the set of vectors satisfying the constraints. Let $t_i, i = 0, 1, \dots, n$, be time values such that $t_i < t_{i+1}, i = 0, 1, \dots, n-1$, without loss of generality we choose $t_0 = 0$, and for later convenience we define $t_{n+1} = +\infty$. Let $(\tilde{x}_i, \tilde{C}_i, \tilde{P}_i)$ be respectively the log-return of the asset price S_i and the prices of European vanilla call and put options on the asset having maturity time $T_i (T_i > t_i)$ and strike price K_i , observed at time $t = t_i, i = 0, 1, \dots, n$. We suppose that the option prices observations $\tilde{C}_i, \tilde{P}_i, i = 0, 1, \dots, n$, are affected by an additive Gaussian error with mean zero and known variance $\phi_i, i = 0, 1, \dots, n$, respectively and that the asset log-returns $\tilde{x}_i, i = 0, 1, \dots, n$, are observed without error. The filtering problem that we consider is the following: Given the vector $\underline{\Theta} \in \mathcal{M}$ find the probability density function $p(x, v_1, v_2, t | \mathcal{F}_t, \underline{\Theta})$, $(x, v_1, v_2) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+, t > 0$, of the random variables $x_t, v_{1,t}$ and $v_{2,t}, t > 0$, solution of (1), (2), (3), (4) conditioned to the observations $\mathcal{F}_t = \{(\tilde{x}_i, \tilde{C}_i, \tilde{P}_i) : t_i \leq t, i > 0\}$, $t > 0$ and forecast the values of the state variables $x_t, v_{1,t}, v_{2,t}$ for $t > 0$ and in particular for $t \neq t_i, i = 0, 1, \dots, n$, and for $t > t_n$.

Let us outline the solution of the filtering problem (see [1] for more details). We define the following functions:

$$f_0(x, v_1, v_2; \underline{\Theta}) = \delta(x - \tilde{x}_0) \delta(v_1 - \tilde{v}_{1,0}) \delta(v_2 - \tilde{v}_{2,0}), (x, v_1, v_2) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+, \quad (6)$$

and for $i = 1, 2, \dots, n$, $(x, v_1, v_2) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+$

$$f_i(x, v_1, v_2; \underline{\Theta}) = \frac{\delta(x - \tilde{x}_i) p_{i-1}(x, v_1, v_2, t_i^- | \underline{\Theta}) \pi_1(x, v_1, v_2, t_i | \underline{\Theta})}{\int_0^{+\infty} \int_0^{+\infty} p_{i-1}(\tilde{x}_i, v'_1, v'_2, t_i^- | \underline{\Theta}) \pi_1(\tilde{x}_i, v'_1, v'_2, t_i^- | \underline{\Theta}) dv'_1 dv'_2}, \quad (7)$$

where $p_i(x, v_1, v_2, t | \underline{\Theta}) = p(x, v_1, v_2, t | \mathcal{F}_t, \underline{\Theta})$, $(x, v_1, v_2) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+, t_i < t < t_{i+1}, i = 0, 1, \dots, n$, is the joint probability density function of the random variables $x_t, v_{1,t}, v_{2,t}$, solution (1)-(4) when $t_i < t < t_{i+1}$ conditioned to the observations made up to time $t = t_i$ and $p_{i-1}(\tilde{x}_i, v_1, v_2, t_i^- | \underline{\Theta}) = \lim_{t \rightarrow t_i^-} p_{i-1}(\tilde{x}_i, v_1, v_2, t; \underline{\Theta})$, $\lim_{t \rightarrow t_i^-}$ means left limit for t that goes to $t_i, i = 0, 1, \dots, n$. Finally $\pi_1(\tilde{x}_i, v_1, v_2, t_i | \underline{\Theta}), (\tilde{x}_i, v_1, v_2) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+, i = 0, 1, \dots, n$, is given by:

$$\pi_1(\tilde{x}_i, v_1, v_2, t_i | \underline{\Theta}) = \frac{1}{\sqrt{2\pi\phi_i}} \frac{1}{\sqrt{2\pi\phi_i}} e^{\left(-\frac{1}{2\phi_i} \left[(\tilde{C}_i - C(\tilde{x}_i, v_1, v_2, t_i; K_i, T_i, \underline{\Theta}))^2 + (\tilde{P}_i - P(\tilde{x}_i, v_1, v_2, t_i; K_i, T_i, \underline{\Theta}))^2 \right] \right)}, \quad (8)$$

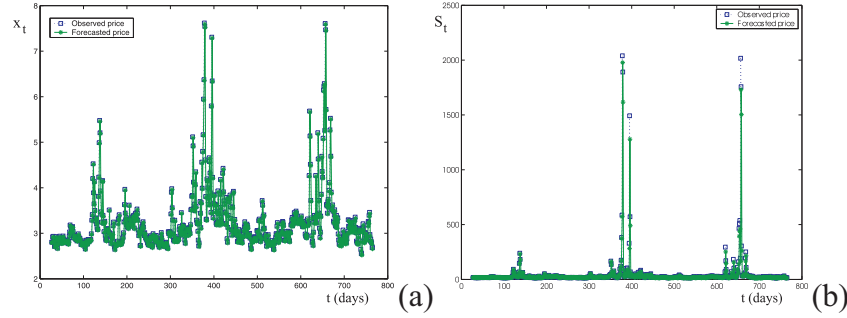
and C and P in (8) are the call and put option price formulae in the model (1)-(4) deduced in [2]. In [1] an integral formula for p_i is derived and a tracking procedure able to forecast the values of the state variables $x_t, v_{i,t}$, $t > 0, i = 1, 2$, is obtained. In [1] the problem of estimating the vector $\underline{\Theta}$ is reduced to the solution of the following maximum likelihood problem:

$$\max_{\underline{\Theta} \in \mathcal{M}} F(\underline{\Theta}) \quad (9)$$

where $F(\underline{\Theta})$ is the (log-) likelihood function defined as follows:

$$F(\underline{\Theta}) = \sum_{i=0}^{n-1} \log \left[\int_0^{+\infty} \int_0^{+\infty} p_i(\tilde{x}_{i+1}, v_1, v_2, t_{i+1}^- | \underline{\Theta}) \pi_1(\tilde{x}_{i+1}, v_1, v_2, t_{i+1} | \underline{\Theta}) dv_1 dv_2 \right] + \log \left[(\tilde{x}_0, \tilde{v}_{1,0}, \tilde{v}_{2,0}, t_0 | \underline{\Theta}) \right] \quad \underline{\Theta} \in \mathcal{M}. \quad (10)$$

Figure 1: *Forecasted values(one day in the future)(stars) and observed values(squares) of the log-returns of electric power prices (a) and of the electric power prices (b).*



In the real data case we use the estimated values of the parameters and of the initial stochastic variances obtained solving the calibration problem to produce through the tracking procedure (i.e. taking the expected values of the state variables conditioned to the observations made extended in the future using the dynamics of the model) forecasts for future times of the electric power prices and of the associated stochastic variances. Figure 1 shows the observations and the forecasts of the log-return and of the corresponding asset (electric power) price one day in the future obtained using in the forecasting procedure the model parameters and the initial stochastic variances obtained solving the maximum likelihood problem that uses as data the observations made in a time window made of 26 consecutive days ending with the price of the day before the forecast. Figure 1a) and Figure 1b) show x_t , $t > 0$, and S_t , $t > 0$, respectively. Figure 1 shows forecasts of very high quality.

References

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