

Sudoku grids. Designs and contingency tables

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Abstract: In this paper we analyze the sudoku games as both fractions of designed experiments and contingency tables. In particular, from the theory of contingency tables we apply the notion of Markov basis as introduced in Diaconis and Sturmfels (1998). In this application the Markov basis is very large and we investigate its structure.

Keywords: Algebraic Statistics, Categorical Data Analysis, Design of Experiments, Markov basis

1. Sudoku grids and designs

In recent years, sudoku has become a very popular game. The objective of the game is to complete a 9×9 grid with the digits from 1 to 9. Each digit must appear once and only once in each column, each row and each of the nine 3×3 boxes. Although the 9×9 is the most popular scheme, in this work we introduce general $p^2 \times p^2$ grids ($p \geq 2$), but then, for computational limits, we restrict ourselves to the case $p = 2$.

The sudoku grids are special cases of Latin squares in the class of *gerechte designs*, see Bailey *et al.* (2008). In Fontana and Rogantin (2009) the connections between sudoku grids and experimental designs are extensively studied in the framework of Algebraic Statistics. We expect that the computations presented here will form a contribution to understand the connections between designed experiments and contingency tables.

A sudoku grid is a particular subset of cardinality $p^2 \times p^2$ of the $p^2 \times p^2 \times p^2$ possible assignments of a digit between 1 and p^2 (or more generally p^2 symbols) to the cells of the grid. Under the point of view of Design of Experiments, a sudoku can be considered as a fraction of a full factorial design with six factors R_1, R_2, C_1, C_2, S_1 and S_2 , each having p levels. In this work we keep our exposition simple using integer coding for the level factors, $\{0, \dots, p-1\}$. The grid is divided into $p \times p$ square boxes. The square box is identified through the factors R_1 and C_1 , named as “the band” and “the stack”, respectively. Then, the cell is identified through the factors R_2 and C_2 , named as “the row within a band” and “the column within a stack”. Finally, given a symbol k between 1 and p^2 , the factors S_1 and S_2 provide the base- p representation of $k-1$. It should be noted that two factors for symbols are introduced only for symmetry of representation.

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As an example, in a 4×4 sudoku, if the symbol 3 (coded with 10, the binary representation of 2) is in the second row within the first band ($R_1 = 0, R_2 = 1$) and in the first column of the second stack ($C_1 = 1, C_2 = 0$), as in the left grid of Figure 1, the corresponding point of the design is $(0, 1, 1, 0, 1, 0)$.

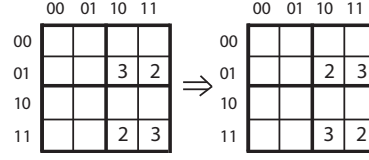


Figure 1: Coding of cells and one move in the class $\mathcal{M}_3$

A sudoku grid, as fraction of a design, is specified through its indicator polynomial function, and a move between two grids is also a polynomial, obtained as difference between the two indicator functions. With such methodology, in Fontana and Rogantin (2009), three classes of moves are described: permutation of symbols, bands, rows within a band, stacks, column within a stack (denoted by $\mathcal{M}_1$), transposition between rows and columns (denoted by $\mathcal{M}_2$) and moves acting on special parts of the sudoku grid (denoted by $\mathcal{M}_3$), as shown on Figure 1.

2. Moves and Markov basis for sudoku grids

The relations between sudoku grids and contingency tables are not trivial. In fact, a sudoku grid is not a contingency table as it contains labels and not counts. To describe a valid sudoku grid as a contingency table, we need to consider a $p \times p \times p \times p \times p \times p$ table $\mathbf{n}$, with 6 indices r_1, r_2, c_1, c_2, s_1 and s_2 , each ranging between 1 and p . The table $\mathbf{n}$ is a 0 – 1 table with $\mathbf{n}_{r_1 r_2 c_1 c_2 s_1 s_2} = 1$ if and only if the (r_1, r_2, c_1, c_2) cell of the grid contains the symbol (s_1, s_2) and is 0 otherwise. This approach has been already sketched in Fontana and Rogantin (2009). A similar approach is also described in Aoki and Takemura (2008) for related applications.

The validity conditions are expressed as follows. Each sudoku grid must have one and only one symbol in each cell, in each row, in each column and in each box. The four constraints translate into the following linear conditions on $\mathbf{n}$.

$$\begin{aligned}
 \sum_{s_1, s_2=1}^p \mathbf{n}_{r_1 r_2 c_1 c_2 s_1 s_2} &= 1 \quad \forall r_1, r_2, c_1, c_2 & \sum_{c_1, c_2=1}^p \mathbf{n}_{r_1 r_2 c_1 c_2 s_1 s_2} &= 1 \quad \forall r_1, r_2, s_1, s_2 \\
 \sum_{r_1, r_2=1}^p \mathbf{n}_{r_1 r_2 c_1 c_2 s_1 s_2} &= 1 \quad \forall c_1, c_2, s_1, s_2 & \sum_{r_2, c_2=1}^p \mathbf{n}_{r_1 r_2 c_1 c_2 s_1 s_2} &= 1 \quad \forall r_1, c_1, s_1, s_2
 \end{aligned}$$

Therefore we have $4p^4$ linear conditions on $\mathbf{n}$ which define a valid sudoku grid. In Diaconis and Sturmfels (1998) the definition of Markov basis is given, as a set of moves to connect any two contingency tables with fixed value of some linear constraints, staying non-negative. Therefore, this approach enables us to generate all the sudoku grids starting from a given one. We refer to Diaconis and Sturmfels (1998) for the formal definition of

Markov basis. At this stage, we simply point out that, given a sudoku $\mathbf{n}$, we are interested in moves $\mathbf{m}$ such that $\mathbf{n} + \mathbf{m}$ or $\mathbf{n} - \mathbf{m}$ is still a sudoku.

Although in many problems involving contingency tables the Markov basis is a small set of moves, easy to compute and to handle, in the case of sudoku grids we have a large number of moves without simple evident symmetries.

3. An example. The 4×4 sudoku

From a theoretical point of view, the computation of a Markov basis guarantees the connectedness of all tables and therefore it gives an actual method to generate all the valid sudoku grids starting from a given one. Nevertheless, the computation of Markov basis needs symbolic computations in polynomial rings with a large number of variables and is unfeasible for our problem, apart from the simplest case $p = 2$, i.e. 4×4 sudoku.

With $p = 2$, using 4ti2 (4ti2 team (2007)), we obtain the Markov basis $\mathcal{M}$. It contains 34,920 elements. Using the set of all the valid sudoku grids that has been reported in Fontana and Rogantin (2009) and that contains 288 different tables and some ad-hoc modules written in SAS-IML (SAS Institute Inc. (2004)), we have explored this Markov basis finding some interesting facts:

- The elements $\mathbf{m}$ of $\mathcal{M}$ that can be summed at least to one sudoku grid, $\mathbf{n}$, in order to obtain again a valid sudoku grid, $\mathbf{n} + \mathbf{m}$, are 2,160, a relatively small subset $\mathcal{M}_f^+$ of $\mathcal{M}$. We observe that $\mathcal{M}_f^-$, that, analogously, is the subset of $\mathcal{M}$ formed by the elements that can be subtracted by at least one sudoku grid to obtain again a valid sudoku grid coincides with $\mathcal{M}_f^+$. Therefore we denote with $\mathcal{M}_f$ the subset of $\mathcal{M}$ that generates $2 \cdot 2,160 = 4,320$ different moves. To provide a term of comparison, we remind that the cardinality of the set of all the differences between two valid sudoku is $288 \cdot 287 = 82,656$ and that we have checked that 78,144 of them are different.
- If we classify each of the 2,160 positive moves generated by $\mathcal{M}_f$ according to both the number of sudoku that can use it and the number of points of the grids that are changed by the move itself, we obtain the following table (an identical result can be obtained for negative moves).

# of Sudoku that can use the move	# of Points moved				Total
	4	8	10	12	
1	0	0	1,536	192	1,728
2	0	336	0	0	336
8	96	0	0	0	96
Total	96	336	1,536	192	2,160

- The 96 moves that change 4 points and can be used 8 times are all the $\mathcal{M}_3$ moves, like the one reported in Figure 1. We have also verified that these moves, together with their negative counterparts, are enough to connect all the sudoku grids.
- The 336 moves that changes 8 points and can be used by 2 tables can be viewed as exchange of symbols, rows and columns. Indeed, for each of the 6 couple of symbols there are 56 of such moves, 40 of them acting on 8 points for a total of 240 moves. Other $4 \cdot 4! = 96$ moves correspond to the exchange of rows and columns. For example, two of them are:

	4	3	
	3		4
4			3
3		4	

 $\Rightarrow$

	3	4	
	4		3
3			4
4		3	

4	2	3	1
3	1	2	4

 $\Rightarrow$

3	1	2	4
4	2	3	1

- The remaining 1, 728 moves appear as composition of the previous ones. For instance, the move

3		4	1
	1	3	2
2	3		4
1	4	2	

 $\Rightarrow$

1		3	4
	3	2	1
3	4		2
2	1	4	

can be seen as the composition of a permutation of two symbols, a permutation of two rows and two moves of type $\mathcal{M}_3$.

We conclude our analysis focusing on regular sudoku fractions. In Fontana and Rogantin (2009), the 288 sudoku fractions have been split into 96 regular fractions and 192 non regular fractions. We have found that, if the sudoku is regular, the number of feasible moves is 14 while, if it is non regular, the number of feasible moves is 26.

4. Conclusion

The use of Markov Basis allowed to study the moves between 4×4 sudoku. However, we agree with A.Takemura and H.Hara (Univ. of Tokyo) that, in a recent presentation, said that “in general, the problem of determining a subset of Markov basis for connecting one particular fiber remains a difficult question”.

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