

Multivariate tail dependence coefficients

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Abstract: The aim of this paper is to give a measure of the tail dependence for n -dimensional Archimedean copula functions. We propose the upper and lower tail dependence coefficients, in a multivariate framework, extending their bivariate definition given in literature.

Keywords: Archimedean copula functions, tail dependence coefficients.

1. Introduction

In order to diversify the risk due to extreme events, a classical problem of the portfolio composition consists into choosing financial assets with a small amount of dependence in situations when extreme negative events occur, on the one hand, and with high dependence in the opposite case. To tackle this problem, it is useful to model the whole dependence structure between variables with a copula function using, as measure of the relationship in the tails of the joint distribution, the upper and lower tail dependence coefficients. The most recent literature on this argument suggests to use the upper and lower tail dependence coefficients to measure the association only between a couple of variables modelled by a bivariate copula function. This does not allow to estimate co-movements among more than two markets or risk factors and other relevant financial variables. For this reason, we propose the expressions of the upper and lower tail dependence coefficients in a multivariate context.

2. Copula and Survival Copula Function

A copula function C is a multivariate distribution function defined on the unit cube $[0, 1]^n$, with uniformly distributed margins (see Joe, 1997 and Nelsen, 2006). The well known Sklar's Theorem shows the relationship between the copula function, C , and the n -dimensional function of the random variables X_i (for $i = 1, \dots, n$), H , with domain $\bar{R}^n$, such that H is grounded, n -increasing and $H(\infty, \dots, \infty) = 1$, defined as

$$H(x_1, \dots, x_n) = P(X_1 \leq x_1, \dots, X_n \leq x_n).$$

According to the theorem, there exists an n -dimensional copula function C such that for all $\mathbf{x}$ in $\bar{R}^n$

$$H(x_1, \dots, x_n) = C(u_1, \dots, u_n) \tag{1}$$

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where u_i is the uniform marginal distribution $F_i(X_i)$. Moreover, by Sklar's theorem, it is possible to derive the relationship between the survival joint function, $\bar{H}(x_1, \dots, x_n)$, and the survival copula function, $\hat{C}(1 - u_1, \dots, 1 - u_n)$. The survival joint function is given by

$$\bar{H}(x_1, \dots, x_n) = P(X_1 > x_1, \dots, X_n > x_n)$$

and it follows from (1) that

$$\bar{H}(x_1, \dots, x_n) = \hat{C}(1 - u_1, \dots, 1 - u_n).$$

For $n = 2$, it has been shown that the survival copula function

$$\begin{aligned} \hat{C}(1 - u_1, 1 - u_2) &= [1 - P(U_1 \leq u_1)] + [1 - P(U_2 \leq u_2)] - [1 - \\ &\quad P(U_1 \leq u_1, U_2 \leq u_2)] \end{aligned} \quad (2)$$

is strictly related to the copula function through the relationship

$$\hat{C}(1 - u_1, 1 - u_2) = 1 - u_1 - u_2 + C(u_1, u_2). \quad (3)$$

Following the same reasoning, we have generalized (2), and hence (3), when C is an Archimedean copula function.

The Archimedean copulas family (see Cherubini *et al.*, 2004) can be built starting from the definition of a generator function $\Phi : [0, 1] \rightarrow R^+$, continuous, decreasing and convex, such that $\Phi(1) = 0$ and where $\Phi^{[-1]}(t)$ is defined the "pseudo-inverse" of $\Phi(t) : \Phi^{[-1]}(t) = \Phi^{-1}(t) \forall t \in [0, \Phi(0)]$ and $\Phi^{[-1]}(t) = 0$ for $t \geq \Phi(0)$. If $\Phi(0) = \infty$, then $\Phi(t)$ and C are said to be strict; if $\Phi(0) < \infty$, $\Phi(t)$ and C are non-strict (for a definition, see Nelsen, 2006). Then an Archimedean copula can be expressed as

$$C(u_1, u_2, \dots, u_n) = \Phi^{-1}(\Phi(u_1) + \dots + \Phi(u_n)).$$

Hence, we have found the relationship, in a multivariate framework, between an Archimedean copula C with generator function Φ , and its related survival copula $\hat{C}$, given by

$$\hat{C}(1 - u_1, \dots, 1 - u_n) = \sum_{i=0}^n \left\{ \binom{n}{n-i} (-1)^i \left[\Phi^{-1} \left(\sum_{j=0}^i \Phi(u_j) \right) \right] \right\}$$

where, in square brackets, $\Phi^{-1}(\Phi(u_0)) = 1$ and $\Phi^{-1}(\Phi(u_1)) = u_1$.

3. Upper and lower tail dependence coefficients

In the literature, the tail dependence is a measure of concordance between less probable values of variables. This concordance tends to concentrate on the lower and upper tails of the joint distribution. In a bivariate context, let $F_i(\cdot)$, $i = 1, 2$, be the marginal distribution functions of two random variables X_1 and X_2 and let u be a threshold value, then the upper tail coefficient, λ_U , is defined as (see e.g. Joe, 1997 and Cherubini *et al.*, 2004)

$$\lambda_U = \lim_{u \rightarrow 1^-} P(F_1(X_1) > u | F_2(X_2) > u) = \lim_{u \rightarrow 1^-} P(U_1 > u | U_2 > u).$$

When $\lambda_U \in (0, 1]$, X_1 and X_2 are asymptotically dependent on the upper tail; if λ_U is null, X_1 and X_2 are asymptotically independent.

Since

$$\begin{aligned} P(U_1 > u | U_2 > u) &= \frac{P(U_1 > u, U_2 > u)}{P(U_2 > u)} \\ &= \frac{1 - P(U_1 \leq u) - P(U_2 \leq u) + P(U_1 \leq u, U_2 \leq u)}{1 - P(U_2 \leq u)}, \end{aligned}$$

then, from (2) and (3) it follows that the upper tail dependence coefficient can be also expressed in terms of copula function, that is

$$\lambda_U = \lim_{u \rightarrow 1^-} \frac{\hat{C}(1-u, 1-u)}{1-u} = \lim_{u \rightarrow 1^-} \frac{1-2u+C(u, u)}{1-u}.$$

In a similar way, the lower tail dependence coefficient, λ_L , is defined as

$$\lambda_L = \lim_{u \rightarrow 0^+} P(F_1(X_1) \leq u | F_2(X_2) \leq u) = \lim_{u \rightarrow 0^+} P(U_1 \leq u | U_2 \leq u)$$

and hence

$$P(U_1 \leq u | U_2 \leq u) = \frac{P(U_1 \leq u, U_2 \leq u)}{P(U_2 \leq u)}.$$

Then, the lower tail dependence coefficient can be expressed in terms of copula,

$$\lambda_L = \lim_{u \rightarrow 0^+} \frac{C(u, u)}{u}.$$

It has been shown (see Embrechts *et al.*, 2003) that for an Archimedean copula each tail dependence coefficient can be derived using the generator function. In fact, if the first derivative, $\Phi^{-1'}(t)$, is finite, then an Archimedean copula does not have upper tail dependence; conversely, λ_U is given by

$$\lambda_U = \lim_{u \rightarrow 1^-} \frac{1-2u+C(u, u)}{1-u} = \lim_{u \rightarrow 1^-} \frac{1-2\Phi^{-1}(\Phi(u)) + \Phi^{-1}(2\Phi(u))}{1-\Phi^{-1}(\Phi(u))}.$$

To solve the limit, it is necessary to apply de L'Hôpital theorem

$$\begin{aligned} \lambda_U &= \lim_{u \rightarrow 1^-} \frac{-2\Phi^{-1'}(\Phi(u))(\Phi'(u)) + \Phi^{-1'}(2\Phi(u))2(\Phi'(u))}{-\Phi^{-1'}(\Phi(u))\Phi'(u)} \\ &= \lim_{u \rightarrow 1^-} \frac{-2\Phi^{-1'}(\Phi(u))(\Phi'(u))}{-\Phi^{-1'}(\Phi(u))\Phi'(u)} + \frac{\Phi^{-1'}(2\Phi(u))2(\Phi'(u))}{-\Phi^{-1'}(\Phi(u))\Phi'(u)} \\ &= 2 - 2 \lim_{u \rightarrow 1^-} \frac{\Phi^{-1'}(2\Phi(u))}{\Phi^{-1'}(\Phi(u))} = 2 - 2 \lim_{t \rightarrow 0} \frac{\Phi^{-1'}(2t)}{\Phi^{-1'}(t)}. \end{aligned} \quad (4)$$

The lower tail dependence coefficient is given by

$$\begin{aligned} \lambda_L &= \lim_{u \rightarrow 0^+} \frac{C(u, u)}{u} = \lim_{u \rightarrow 0^+} \frac{\Phi^{-1}(2\Phi(u))}{\Phi^{-1}(\Phi(u))} = \lim_{u \rightarrow 0^+} \frac{\Phi^{-1'}(2\Phi(u))(2\Phi'(u))}{\Phi^{-1'}(\Phi(u))\Phi'(u)} \\ &= 2 \lim_{t \rightarrow 0} \frac{\Phi^{-1'}(2t)}{\Phi^{-1'}(t)}. \end{aligned} \quad (5)$$

The solutions (Joe, 1997) of (4) and (5) for a widely popularized copula function, the Joe-Clayton or BB7 copula, with generator function $\Phi(t) = (1 - (1 - t)^\theta)^{-\kappa} - 1$, lead to

$$\lambda_U = 2 - 2^{\frac{1}{\theta}} \quad \text{and} \quad \lambda_L = 2^{-\frac{1}{\kappa}}.$$

While the literature focuses on the tail dependence measures for the bivariate copula, we have proposed to extend this concept in a multivariate framework and we have found the expressions of upper and lower tail dependence coefficients. In particular, we have defined the multivariate upper tail dependence coefficient as

$$\begin{aligned} \lambda_U &= \lim_{u \rightarrow 1^-} P(F_1(X_1) > u | F_2(X_2) > u, \dots, F_n(X_n) > u) \\ &= \lim_{u \rightarrow 1^-} \frac{\hat{C}_n(1 - u, \dots, 1 - u)}{\hat{C}_{n-1}(1 - u, \dots, 1 - u)} \\ &= \lim_{u \rightarrow 1^-} \frac{\sum_{i=0}^n \left\{ \binom{n}{n-i} (-1)^i [\Phi^{-1}(i \cdot \Phi(u))] \right\}}{\sum_{i=0}^{n-1} \left\{ \binom{n-1}{n-i-1} (-1)^i [\Phi^{-1}(i \cdot \Phi(u))] \right\}}. \end{aligned}$$

Solving this limit for the Joe-Clayton copula, we have obtained

$$\lambda_U = \frac{\sum_{i=0}^n \left[\binom{n}{n-i} (-1)^i (i)^{\frac{1}{\kappa}} \right]}{\sum_{i=0}^{n-1} \left[\binom{n-1}{n-i-1} (-1)^i (i)^{\frac{1}{\kappa}} \right]}.$$

Furthermore, we have derived the multivariate lower tail dependence coefficient as

$$\begin{aligned} \lambda_L &= \lim_{u \rightarrow 0^+} P(F_1(X_1) \leq u | F_2(X_2) \leq u, \dots, F_n(X_n) \leq u) \\ &= \lim_{u \rightarrow 0^+} \frac{C_n(u, \dots, u)}{C_{n-1}(u, \dots, u)} = \lim_{u \rightarrow 0^+} \frac{[\Phi^{-1}(n \cdot \Phi(u))]}{[\Phi^{-1}((n-1) \cdot \Phi(u))]} \end{aligned}$$

The solution of this limit for the Joe-Clayton copula is

$$\lambda_L = \left(\frac{n}{n-1} \right)^{-\frac{1}{\theta}}.$$

4. Concluding Remarks

These results can be interestingly applied to measure the extreme co-movements among financial variables. If $X_1, \dots, X_n$ denote financial returns of n assets, then the multivariate upper (lower) tail dependence coefficient is the probability that an asset assumes very high (low) values given very high (low) values for remaining $n - 1$ assets.

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